2/23/2008 Linear antenna:

$$I(z) = I_0 \sin\left[k\left(\frac{d}{2} - |z|\right)\right]$$ symmetric. vanishes @ $z = \pm \frac{d}{2}$.

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{2} \cdot Z_0 |I_0|^2 \cdot \frac{1}{4\pi^2} \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]^2}$$

cf. dipole:

(1) $kd \ll 1$. $\left(\cancel{1} - \frac{1}{2}\left(\frac{kd}{2}\cos\theta\right)^2\right) - \left(\cancel{1} - \frac{1}{2}\left(\frac{kd}{2}\right)^2\right)$

$$= \frac{1}{2}\left(\frac{kd}{2}\right)^2 (1 - \cos^2\theta) = \frac{1}{8}(kd)^2 \sin^2\theta.$$

$$\frac{dP}{d\Omega} \to \frac{1}{2} \cdot Z_0 |I_0|^2 \cdot \frac{1}{4\pi^2} \cdot \frac{1}{64} \cdot (kd)^4 \frac{\sin^4\theta}{\sin^2\theta}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{2} Z_0 |I_0|^2 \cdot \frac{(kd)^4}{256\pi^2} \cdot \sin^2\theta}$$

Looks like dipole.

2 dipole moment:

$$\frac{\partial \lambda}{\partial t} + \frac{\partial I}{\partial z} = -i\omega\lambda \mp I_0 k \cdot \cos\left[k\left(\frac{d}{2} - |z|\right)\right]$$

$$\lambda = \pm i \frac{I_0}{c} \cos\left[k\left(\frac{d}{2} - |z|\right)\right]$$

$$\vec{p} = \int_{-d/2}^{d/2} \hat{z}\, dz' \cdot \lambda = 4i \frac{I_0}{c} d^2 \frac{\sin^2\left(\frac{kd}{4}\right)}{(kd)^2} \hat{z}$$

$kd \ll 1$.

$$\boxed{\vec{p} \to \frac{i I_0 d^2}{4c} \hat{z}}$$

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} c^2 Z_0 k^4 \left|\hat{r} \times \vec{p}\right|^2$$

$$= \frac{1}{32\pi^2} \cancel{c^2} Z_0 k^4 \cdot \frac{|I_0|^2 d^4}{16\, \cancel{c^2}}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{2} \cdot Z_0 |I_0|^2 \frac{(kd)^4}{256\pi^2} \sin^2\theta}$$

$$\boxed{P = \frac{1}{2} Z_0 |I_0|^2 \cdot \frac{(kd)^4}{96\pi}}$$

$P(kd)$

Multipole expansion.

$(\nabla^2 + k^2)\psi = -j$ $= 0$ outside sources.

Helmholtz.

$\psi(r,\theta,\phi,t) = R(r)\, Y_{lm}(\theta,\phi)\, e^{-i\omega t}$

$$\rightarrow \boxed{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2} + \frac{\omega^2}{c^2} = 0}$$

Almost. Bessel's equation. $\frac{1}{r}, \frac{2}{r}$ $\omega^2, l(l+1)$

Behaviors: $k = \frac{\omega}{c}$ $R'' + k^2 R = 0$ $R \sim \sin kR, \cos kR$

$kr \gg 1$.

$R = r^p \sin kr$ $R' = p r^{p-1} \sin kr + k r^p \cos kr$

$$R'' = p(p-1) r^{p-2} \sin kr + 2pk r^{p-1} \cos kr - k^2 r^p \sin kr.$$

$$\left(p(p-1) \cancel{r^{p-2}} \sin kr + 2pk r^{p-1} \cos kr - k^2 \cancel{r^p \sin kr} \right)$$
$$+ \frac{2}{r}\left(p \cancel{r^{p-1}} \sin kr + k r^p \cos kr \right) + \cancel{k^2 r^p \sin kr} = 0$$

$(p+1)(2k r^{p-1} \cos kr) = 0$ $p = -1$ $\frac{\sin kr}{r}, \frac{\cos kr}{r}.$

8

Small kr. $R'' + \frac{2}{r}R' - \frac{l(l+1)}{r^2}R = 0$

$R \sim r^p$ $\quad p(p-1) + 2p - l(l+1) = 0$

$$p(p+1) = l(l+1) \qquad \begin{array}{l} p = l \\ p = -(l+1) \end{array}$$

(static).

Let $R = u/\sqrt{r}$ (lucky guess)

$$R' = \frac{u'}{\sqrt{r}} - \frac{1}{2}\frac{u}{r^{3/2}}$$

$$R'' = \frac{u''}{r^{1/2}} - 2\cdot\frac{1}{2}\cdot\frac{u'}{r^{3/2}} + \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\frac{u}{r^{5/2}}$$

$$= \frac{u''}{r^{1/2}} - \frac{u'}{r^{3/2}} + \frac{3}{4}\frac{u}{r^{5/2}}$$

$$\left(\frac{u''}{r^{1/2}} - \frac{u'}{r^{3/2}} + \frac{3}{4}\frac{u}{r^{5/2}}\right) + \frac{2}{r}\left(\frac{u'}{r^{1/2}} - \frac{1}{2}\frac{u}{r^{3/2}}\right)$$

$$+ \left(k^2 - \frac{l(l+1)}{r^2}\right)\frac{u}{r^{1/2}} = 0$$

(5)

$$\frac{1}{r^{1/2}}\left[u'' + \frac{u'}{r}(2-1) + \frac{u}{r^2}\left(\frac{3}{4}-1\right) + \left(k^2 - \frac{l(l+1)}{r^2}\right)u\right] = 0.$$

$$u'' + \frac{1}{r}u' + \left(k^2 - \frac{(l+\frac{1}{2})^2}{r^2}\right)u = 0$$

$$\underline{u = J_{l+\frac{1}{2}}(kr)}, \quad \underline{N_{l+\frac{1}{2}}(kr)}$$

$$R = u/\sqrt{r}.$$

<u>let</u>

$$j_l(x) = \sqrt{\frac{\pi}{2k}}\; J_{l+\frac{1}{2}}(x)$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}}\; N_{l+\frac{1}{2}}(x)$$

<u>Spherical Bessel Functions</u>