2/26/2018 Green's function for Helmholtz equation

$$(\nabla^2+k^2)\,G(\vec{x},\vec{x}') = -\delta^{(3)}(\vec{x}-\vec{x}')$$ 9.93

$$G = \sum_{l,m} f_l(kr)\,Y_{lm}.$$

$r\to 0$ nonsingular. $j_l(kr) \sim r^l$

$r\to\infty$ outgoing waves. $h_l^{(1)}(kr) \sim \frac{e^{ikr}}{r}.$

$$G_< = \sum_{l,m} A^<_{lm}\, j_l(kr)\, Y_{lm}(\theta,\phi) \qquad r<r'$$

$$G_> = \sum_{lm} A^>_{lm}\, h_l^{(1)}(kr)\, Y_{lm}(\theta,\phi) \qquad r>r'$$

$r=r'$ $(\hat{r}\neq\hat{r}')$. continuous.

$$\Rightarrow\quad A^<_{lm}\, j_l(kr') = A^>_{lm}\, h_l^{(1)}(kr') = C_{lm}\, j_l\, h_l^{(1)}\, Y^*$$

$$G = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}\, j_l(kr_<)\, h_l^{(1)}(kr_>)$$

$$\times\, Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

past experience

$$\int_{r'-\epsilon}^{r'+\epsilon} dr \left(\nabla^2+k^2\right) G = -\int_{r'-\epsilon}^{r'+\epsilon} dr' \, \frac{\delta(r-r')}{r'^2} \, \delta(\Omega+\Omega')$$

$$= -\frac{1}{r'^2} \underline{\sum_{lm} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)}$$

$$= \sum_{l,m} C_{lm} \int_{r'-\epsilon}^{r'+\epsilon} dr \left(\frac{\partial^2}{\partial r^2} + \overset{O(\epsilon)}{\cancel{\frac{2}{r}\frac{\partial}{\partial r}}} + \overset{O(\epsilon)}{\cancel{k^2}} - \overset{O(\epsilon)}{\cancel{\frac{l(l+1)}{r'^2}}} \right)$$
$$\times \left[j_l(kr_<) h_l^{(1)}(kr_>) Y^{*'}_{lm} Y_{lm} \right]$$

$$= \sum_{lm} C_{lm} \frac{\partial}{\partial r}\left[j_l(kr_<) h_l^{(1)}(kr_>) \right]\Big|_{r=r'} Y^{*'}_{lm} Y_{lm}$$

Term-by-term

$$C_{lm} \frac{\partial}{\partial r}\left[j_l(kr_<) h_l^{(1)}(kr_>) \right]\Big|_{r=r'} = -\frac{1}{r'^2}$$

$$r = r'+\epsilon \Rightarrow r_> \qquad r = r'-\epsilon = r_<$$

$$= C_{lm} \left[\frac{\partial}{\partial r}\left(j_l(kr') h_l^{(1)}(kr) \right) - \frac{\partial}{\partial r}\left(j_l(kr) h_l^{(1)}(kr') \right) \right]$$

$$= C_{lm} k \left[j_l h_l' - j_l' h_l \right]\Big|_{x=kr'} = \frac{1}{r'^2}$$

③

let $W = jh' - hj'$. $\quad f'' + \frac{2}{x} f' + \left(1 - \frac{l(l+1)}{x^2}\right) f = 0.$

$$\frac{dW}{dx} = j \cdot h'' - h \cdot j''$$

$$= j\left[\left(-\frac{2}{x} h' - \cancel{\left(1 - \frac{l(l+1)}{x^2}\right) h}\right] - h\left(-\frac{2}{x} j' - \cancel{\left(1 - \frac{l(l+1)}{x^2}\right) j}\right)$$

$$= -\frac{2}{x}\left(jh' - hj'\right) = -\frac{2}{x} W.$$

$$\frac{d}{dx}(x^2 W) = x^2 \cdot \frac{dW}{dx} + 2xW = x^2\left(-\frac{2W}{x}\right) + 2xW = 0$$

$$x^2 W = \text{constant}$$

(3.89)

(9.88) $\quad (x \to 0) \quad j_l \to \frac{x^l}{(2l+1)!!}$

$$h_l = j_l + i n_l \to \cancel{\frac{x^l}{(2l+1)!!}} + i\left(-\frac{(2l-1)!!}{x^{l+1}}\right)$$

(3.91)

(9.89) $\quad jh' - hj' \to (-i)\left(\frac{(2l-1)!!}{(2l+1)!!}\right)\left[x^l\left(-\frac{(l+1)}{x^{l+2}}\right) - \frac{1}{x^{l+1}} \cdot l x^{l-1}\right)$

$$= (-i)\frac{1}{(2l+1)}\left[\left(-\frac{l+1}{x^2} - \frac{l}{x^2}\right] = \frac{i}{x^2}\right.$$

$$W = \frac{i}{x^2}$$

$$\boxed{c_{lm} \cdot k \cdot \frac{i}{(kr')^2} = -\frac{1}{r'^2}} \qquad c_{lm} = ik$$

$$\boxed{G = \frac{1}{4\pi} \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} ik \, j_l(kr_<) \, h_l^{(1)}(kr_>) \times Y_{lm}^*(\theta',\phi') \, Y_{lm}(\theta,\phi)}$$

(9.98) $\cancel{\ast\ast\ast}$

$k \to 0$.

$$j_l \to \frac{(kr_<)^l}{(2l+1)!!}$$

$$h_l \to i\left(-\frac{(2l-1)!!}{(kr_>)^{l+1}}\right)$$

$$ikj_<h_> \to \frac{r_<^l}{r_>^{l+1}} \frac{1}{2l+1}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') \, Y_{lm}(\theta,\phi)$$

$r' \to \text{big}$ $\quad j_\ell \to \dfrac{\sin(x - \frac{\ell\pi}{2})}{x}$

$$n_\ell \to \frac{-\cos(x - \ell\frac{\pi}{2})}{x}$$

3.90
~~3.91~~

$$h_\ell^{(1)} \to -i\left(\frac{\cos(x-\frac{\ell\pi}{2}) + i\sin(x - \frac{\ell\pi}{2})}{x}\right)$$

$$= (-i)\cdot \frac{e^{i(x-\ell\pi/2)}}{x} = (-i)^{\ell+1}\frac{e^{ix}}{x}.$$

$$G = \frac{1}{4\pi}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \to \frac{e^{ikr'}}{4\pi r'}e^{-ik\hat{r}\cdot\vec{x}}$$

$$= \sum_{\ell,m} ik\, j_\ell(kr)(-i)^{\ell+1}\frac{e^{ikr'}}{kr'}\, Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

$$= \sum_{\ell m}(-i)^\ell j_\ell(kr)\frac{e^{ikr'}}{r'}\, Y^*_{\ell m}\, Y_{\ell m}.$$

Take: $k\hat{r} = \vec{k}$ $\qquad$ Take c.c.

$$\boxed{e^{i\vec{k}\cdot\vec{x}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} 4\pi\, i^\ell\, j_\ell(kr)\, Y^*_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{r})}$$

(10.43)

could proceed. $A_i, \Phi \to \sum f_\ell(kr) Y_{\ell m}$.

works, sort of, but: $\exists$ two polarizations $\vec{\varepsilon}_1 \vec{\varepsilon}_2$, $\varepsilon_\pm$

$\hat{\nabla} \times \hat{E} = -\frac{\partial \vec{B}}{\partial t} \ldots$

§9.7

Formalism. two modes $\to$ radial E, radial H

Casimir KSH

Maxwell. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = +i\omega \vec{B} = i\omega \mu_0 \vec{H} = ikZ_0 \vec{H}$

$$\vec{\nabla} \times \hat{H} = \frac{\partial \hat{D}}{\partial t} = -i\omega \varepsilon_0 \vec{E} = -ik \frac{\vec{E}}{Z_0}.$$

$$\boxed{\vec{\nabla} \cdot \hat{E} = 0} \quad \boxed{\vec{\nabla} \cdot \hat{H} = 0} \quad \boxed{\vec{\nabla} \times \hat{E} = ikZ_0 \vec{H}} \quad \boxed{\vec{\nabla} \times \vec{H} = -ik \frac{\vec{E}}{Z_0}}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 \vec{E} = \vec{\nabla} \times (ikZ_0)\vec{H} = (ikZ_0)\left(-ik\frac{\vec{E}}{Z_0}\right) = -k^2 \vec{E}$$

$$\boxed{(\nabla^2 + k^2)\vec{E} = 0}$$

Small Theorem:

$$\nabla^2 (\vec{r} \cdot \vec{V}) = \nabla_j \nabla_j (x_i V_i)$$

$$= \nabla_j (\delta_{ij} V_i + x_i \nabla_j V_i)$$

$$= \nabla_i V_i + \delta_{ij} \nabla_j V_i + x_i \nabla_j \nabla_j V_i$$

$$\boxed{\nabla^2 (\vec{r} \cdot \vec{V}) = 2(\vec{\nabla} \cdot \vec{V}) + \vec{r} \cdot (\nabla^2 \vec{V})}$$

$$\nabla^2(\vec{r}\cdot\vec{E}) = \cancel{2(\vec{\nabla}\cdot\vec{E})} + \vec{r}\cdot(\nabla^2\vec{E}) = \vec{r}\cdot(-k^2\vec{E})$$

$$\boxed{(\nabla^2+k^2)(\vec{r}\cdot\vec{E})=0} \qquad \boxed{(\nabla^2+k^2)(\vec{r}\cdot\vec{H})=0}$$

$\underline{\vec{r}\cdot\vec{E}} = \sum f_\ell(kr)\,Y_{\ell m}(\theta,\phi)$ electric multipole modes

$\vec{r}\cdot\vec{H} = \sum g_\ell(kr)\,Y_{\ell m}(\theta,\phi).$ magnetic multipole modes

For full fields: need. $\boxed{\vec{L} = \frac{1}{i}\,\vec{r}\times\vec{\nabla}}$ (9.101)

Ch=1 $\vec{L}$ dimensionless

$$\vec{L} = \frac{1}{i}(r\hat{r})\times\left(\cancel{\hat{r}\frac{\partial}{\partial r}} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\boxed{\vec{L} = \frac{1}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)}$$

lives in θ-ϕ space: $\hat{\theta}, \hat{\phi}$ components
$\partial_\theta, \partial_\phi$ derivatives

$$\boxed{\hat{r}\cdot\vec{L}=0} \quad \boxed{\vec{L}\,f(r)=0} \quad \boxed{\frac{\partial\vec{L}}{\partial r}=0}$$

$$\boxed{[\vec{L}, r]=0} \quad \boxed{\left[\vec{L}, \frac{\partial}{\partial r}\right]=0} \quad [\vec{L}, \hat{r}]\neq 0$$