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2/28/2018 could proceed. $A_i, \Phi \to \sum f_\ell(kr) Y_{\ell m}$.

works, sort of, but: $\exists$ two polarizations $\vec{\varepsilon}_1, \vec{\varepsilon}_2$, $\vec{\varepsilon}_\pm$

$\vec{\nabla} \times \hat{E} = -\frac{\partial \vec{B}}{\partial t}$ . . .

Casimir 454

§9.7

Formalism. two modes $\to$ radial $\vec{E}$, radial $\vec{H}$!

Maxwell. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = +i\omega \vec{B} = i\omega \mu_0 \vec{H} = ik Z_0 \vec{H}$

$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega \varepsilon_0 \vec{E} = -ik \frac{\vec{E}}{Z_0}$.

$\boxed{\vec{\nabla} \cdot \vec{E} = 0}$ $\boxed{\vec{\nabla} \cdot \vec{H} = 0}$ $\boxed{\vec{\nabla} \times \vec{E} = ik Z_0 \vec{H}}$ $\boxed{\vec{\nabla} \times \vec{H} = -ik \frac{\vec{E}}{Z_0}}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 \vec{E} = \vec{\nabla} \times (ik Z_0) \vec{H} = (ik Z_0)\left(-ik \frac{\vec{E}}{Z_0}\right) = -k^2 \vec{E}$

$\boxed{(\nabla^2 + k^2)\vec{E} = 0}$

Small Theorem:

$\nabla^2(\vec{r} \cdot \vec{V}) = \nabla_j \nabla_j (x_i V_i)$

$= \nabla_j (\delta_{ij} V_i + x_i \nabla_j V_i)$

$= \nabla_i V_i + \delta_{ij} \nabla_j V_i + x_i \nabla_j \nabla_j V_i$

$$\boxed{\nabla^2(\vec{r} \cdot \vec{V}) = 2(\vec{\nabla} \cdot \vec{V}) + \vec{r} \cdot (\nabla^2 \vec{V})}$$

(2)

$$\nabla^2(\vec{r}\cdot\vec{E}) = 2(\cancel{\nabla\cdot\vec{E}}) + \vec{r}\cdot(\nabla^2\vec{E}) = \vec{r}\cdot(-k^2\vec{E})$$

$$\boxed{(\nabla^2+k^2)(\vec{r}\cdot\vec{E})=0} \quad \boxed{(\nabla^2+k^2)(\vec{r}\cdot\vec{H})=0}$$

$\vec{r}\cdot\vec{E} = \sum f_\ell(kr)\, Y_{\ell m}(\theta,\phi)$ electric multipole modes

$\vec{r}\cdot\vec{H} = \sum g_\ell(kr)\, Y_{\ell m}(\theta,\phi)$. magnetic multipole modes

For full fields: need. $\boxed{\vec{L} = \frac{1}{i}\vec{r}\times\nabla}$ (9.101)

($\hbar=1$) $\vec{L}$ dimensionless

$$\vec{L} = \frac{1}{i}(r\hat{r})\times\left(\cancel{\hat{r}\frac{\partial}{\partial r}} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\boxed{\vec{L} = \frac{1}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)}$$

lives in θ-ϕ space: $\hat{\theta},\hat{\phi}$ components

$\partial_\theta, \partial_\phi$ derivatives

$$\boxed{\hat{r}\cdot\vec{L}=0} \quad \boxed{\vec{L}f(r)=0} \quad \boxed{\frac{\partial\vec{L}}{\partial r}=0}$$

$$\boxed{[\vec{L}, r]=0} \quad \boxed{\left[\vec{L}, \frac{\partial}{\partial r}\right]=0} \quad [\vec{L}, \hat{r}] \neq 0$$

$$\vec r \times \vec L = \vec r \times \left(\frac{1}{i}\vec r \times \vec\nabla\right)$$

$$= \frac{1}{i}\left[\vec r(\vec r\cdot\vec\nabla) - (\vec r\cdot\vec r)\vec\nabla\right]$$

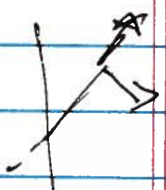
$$= \frac{r^2}{i}\left[\hat r\frac{\partial}{\partial r} - \vec\nabla\right]$$

$$\boxed{\vec\nabla = \hat r\frac{\partial}{\partial r} - \frac{i}{r}\hat r\times\vec L}$$

$$\boxed{L_z = -i\frac{\partial}{\partial\phi}}$$

$L_z Y_{lm} = m Y_{lm}$

$L^2 Y_{lm} = l(l+1) Y_{lm}$.



$\hat z = \hat r\cos\theta - \hat\theta\sin\theta$

$$\hat z\cdot\vec L = (-\hat\theta\sin\theta)\cdot\left(-\frac{1}{i}\frac{\hat\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right) = -i\frac{\partial}{\partial\phi}$$

"Electric multipole" $E_r \neq 0$, "TM mode"

$$\vec{r}\cdot\vec{E}^{E}_{lm} = -\frac{Z_0\, l(l+1)}{k} f_l(kr)\, Y_{lm}(\theta,\phi)$$

$$(\nabla^2 + k^2)(\vec{r}\cdot\vec{E}) = 0$$

$$\left(\hat{r}\cdot\vec{H}^{E}_{lm} = 0\right)$$

$$\vec{L}\cdot\vec{H}^{E}_{lm} = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{H} = \frac{1}{i}\vec{r}\cdot(\vec{\nabla}\times\vec{H})$$

$$= \frac{1}{i}\vec{r}\cdot\left(\frac{-ik\vec{E}}{Z_0}\right) = -\frac{k}{Z_0}(\vec{r}\cdot\vec{E})$$

$$= -\frac{k}{Z_0}\left(-\frac{Z_0\, l(l+1)}{k}\right) f_l(kr)\, Y_{lm}$$

$$\vec{L}\cdot\vec{H}^{E}_{lm} = l(l+1) f_l Y_{lm}$$

$$\vec{H}^{E}_{lm} = f_l(kr)\,\vec{L}\,Y_{lm}$$

$$\vec{E}^{E}_{lm} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}^{E}_{lm}$$

"magnetic multipole" mode.

$(\vec{r}\cdot\vec{H}^{m}_{lm})\neq 0$ $\qquad$ $[\vec{r}\cdot\vec{E}^{m}_{lm}=0$, "TE mode"$]$

$$\vec{r}\cdot\vec{H} = \frac{l(l+1)}{k} g_l(kr)\, Y_{lm}$$

$$\vec{L}\cdot\vec{E}^{m}_{lm} = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{E} = \frac{1}{i}\vec{r}\cdot(\vec{\nabla}\times\vec{E})$$

$$= \frac{1}{i}\vec{r}\cdot\left(ikZ_0\vec{H}\right) = kZ_0(\vec{r}\cdot\vec{H})$$

$$= Z_0\, l(l+1)\, g(kr)\, Y_{lm}$$

$$\vec{E}^{m}_{lm} = Z_0\, g_l(kr)\, \vec{L} Y_{lm}$$

$$\vec{H}^{m}_{lm} = \frac{-i}{kZ_0}\vec{\nabla}\times\vec{E}^{m}_{lm}$$

$$\boxed{\frac{\vec{L}Y_{lm}}{\sqrt{l(l+1)}} = \vec{X}_{lm}}$$

vector spherical harmonic.

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$(l,m) = (1,0) \qquad \underline{Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta}.$

$$\vec{X}_{10} = \frac{\vec{L} Y_{10}}{\sqrt{1\cdot 2}} = \frac{1}{\sqrt{2}} \sqrt{\frac{3}{4\pi}} \frac{1}{i} \hat{\phi} (-\sin\theta) = i \sqrt{\frac{3}{8\pi}} \hat{\phi} \sin\theta$$

$$\boxed{|\vec{X}_{10}|^2 = \frac{3}{8\pi} \sin^2\theta}$$

$\hat{r} \times \vec{m} \to \hat{r} \times \hat{z} = \sin\theta \, \hat{\phi}$

$(\hat{r} \times \hat{p}) \times \hat{r} \quad \sim \quad \sin\theta \, \hat{\theta}. \quad \hat{r} \times \vec{X}_{10}.$

$$\vec{X}_{1,1} = \frac{1}{\sqrt{2}} \left(-\sqrt{\frac{3}{8\pi}}\right) \frac{1}{i} \left(\hat{\phi} \cdot \cos\theta - \frac{\hat{\theta}}{\cancel{\sin\theta}} \cancel{\sin\theta} \cdot i\right) e^{i\phi}$$

$$\boxed{|\vec{X}_{1,1}|^2 = \frac{3}{16\pi} (1 + \cos^2\theta)}$$

$\hat{r} \times (\hat{x} + i\hat{y})$

$= e^{i\phi} (-i\hat{\theta} + \cos\theta \, \hat{\phi})$

$= -i \, (\hat{\theta} + i \cos\theta \, \hat{\phi}) e^{i\phi}$

$P_l^1(\cos\theta)$

$$\vec{X}_{l,0} = \frac{1}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \frac{1}{i} \hat{\phi} (-\sin\theta) P_l'(\cos\theta).$$

$$|\vec{X}_{l,0}|^2 = \frac{2l+1}{4\pi l(l+1)} \sin^2\theta \left[P_l'(\cos\theta)\right]^2$$

$\sin^l\theta \, e^{il\phi}$

$$\vec{X}_{l,l} \sim \hat{\phi} \left[l (\sin\theta)^{l-1} \cdot \cos\theta - \frac{\hat{\theta}}{\sin\theta} \cdot i \, l \cdot \sin^l\theta \right] e^{il\phi}$$

$$|\vec{X}_{l,l}|^2 \sim \sin^{2(l-1)}\theta \cdot (1 + \cos^2\theta)$$

$$\int d\Omega\, \vec{X}^{*}_{lm} \cdot \vec{X}_{l'm'} = \frac{1}{\sqrt{l(l+1)}}\frac{1}{\sqrt{l'(l'+1)}} \int d\Omega\, (\vec{L}Y_{lm})^{*} \cdot (\vec{L}Y_{l'm'})$$

integrate ϕ by parts $\rightarrow$ (-1). (periodic no surface term

integrate θ by parts. $(-)$ $\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\cdot\sin\theta\frac{\partial}{\partial\theta}}$ $\rightarrow L^2$

$\left(\frac{1}{i}\right)^{*} \rightarrow -\left(\frac{1}{i}\right)$.

$$\int d\Omega\, (\vec{L}F)^{*} G = \int d\Omega\, F^{*}(\vec{L}G)$$

$$\int d\Omega\, \vec{X}^{*}_{lm} \cdot \vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\frac{1}{\sqrt{l'(l'+1)}} \int d\Omega\, Y^{*}_{lm} L^2 \underline{Y_{l'm'}}.$$

$$= \frac{l(l+1)}{\sqrt{l(l+1)}\sqrt{l'(l'+1)}}\, \delta_{ll'} \cdot \delta_{mm'} = \delta_{ll'}\,\delta_{mm'}$$

$$\int d\Omega\, (\hat{r}\times\vec{X})^{*} \cdot (\hat{r}\times\vec{X}') \rightarrow \text{same}$$

$$\hat{r}\cdot\left(\vec{X}^{*}\times(\hat{r}\times\vec{X}')\right) = \hat{r}\cdot\left(\hat{r}\, \vec{X}^{*}\cdot\vec{X}' - \cancel{\vec{X}'(\hat{r}\cdot\vec{X}^{*})}\right)$$

$$= (\hat{r}\cdot\hat{r})(\vec{X}^{*}\cdot\vec{X}').$$

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$[\vec{L}, s] = 0.$

$[L_i, V_j] = i\varepsilon_{ijk} V_k.$ $\quad [V_i, L_j] = -[L_j, V_i]$

$= -i\varepsilon_{jik} V_k = +\varepsilon_{ijk} V_k.$

$\vec{r}$ is a vector.
$\vec{\nabla}$ is a vector

$\vec{L} = \frac{1}{i} \hat{r} \times \hat{\nabla}$ is a vector.

$\boxed{[L_i, L_j] = i\varepsilon_{ijk} L_k}$

$\boxed{\vec{L} \times \vec{L} = i\vec{L}}$

$\vec{L} \cdot (\hat{r} \times \vec{L}) = \varepsilon_{ijk} L_i \hat{r}_j L_k$

$= \varepsilon_{ijk} \left(L_i \cancel{L_k} \hat{r}_j + i\varepsilon_{jkm} \hat{r}_m \right)$

$= \underbrace{(\varepsilon_{ijk} L_i L_k)}_{\vec{L} \times \vec{L} = i\vec{L}} \hat{r}_j + i \underbrace{[\varepsilon_{ijk} \varepsilon_{jkm}]}_{2\delta_{im}} L_i \hat{r}_m$

$\cancel{L_j \hat{r}_j} \to 0 \qquad \vec{L} \cdot \cancel{\hat{r}} = 0$

$\boxed{\int d\Omega \, \vec{Y}^*_{\ell m} \cdot (\hat{r} \times \vec{Y}_{\ell' m'}) = 0}$