3/2/2018. From pure thought.

$$\vec{H} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[a^E_{lm} \cdot f_l(kr) \vec{X}_{lm} - \frac{i}{k} a^M_{lm} \vec{\nabla} \times \left(g_l(kr) \vec{X}_{lm} \right) \right]$$

$$\vec{E} = Z_0 \sum_{lm} \left[\frac{i}{k} a^E_{lm} \vec{\nabla} \times \left(f_l \vec{X}_{lm} \right) + a^M_{lm} \cdot g_l(kr) \vec{X}_{lm} \right]$$

$\vec{\nabla} \cdot \left(f(r) \vec{X}_{lm} \right) = 0.$ (Hw).

$$\vec{\nabla} \times \left(\vec{\nabla} \times f\vec{X} \right) = \vec{\nabla} \left(\cancel{\vec{\nabla} \cdot (f\vec{X})} \right) - \nabla^2 \left(f\vec{X} \right) = +k^2 f\vec{X}.$$

$[\vec{L}, \nabla^2] = 0$ Hw.

$\vec{\nabla} \cdot \vec{E} = 0$ $\quad$ $\vec{\nabla} \cdot \vec{H} = 0$ ✓

$(\nabla^2 + k^2)\vec{E} = 0$ $\quad$ $(\nabla^2 + k^2)\vec{H} = 0$ ✓

$\vec{\nabla} \times \vec{E} = ik Z_0 \vec{H}$ $\quad$ $\vec{\nabla} \times \vec{H} = \dfrac{-ik}{Z_0} \vec{E}$ ✓

$$\vec{r} \cdot \vec{H} = \sum_{lm} \left(a^E_{lm} f_l \cancel{\vec{r} \cdot \vec{X}_{lm}} - \frac{i}{k} a^M_{lm} \vec{r} \cdot (\vec{\nabla} \times g\vec{X}) \right)$$

$$= \sum_{lm} \left(\frac{-i}{k} a^M_{lm} \right) (\vec{r} \times \vec{\nabla}) \cdot g\vec{X}$$

$$= \sum_{lm} \frac{1}{k} a_{lm} \vec{L} \cdot (g\vec{X}) = \sum_{lm} \frac{\sqrt{l(l+1)}}{k} a^M_{lm} \, g Y_{lm}.$$

$$\int d\Omega\, \vec{X}^*_{lm} \cdot \vec{X}_{l'm'} = \underline{\delta_{ll'}\,\delta_{mm'}} \qquad \Bigg| \quad \int d\Omega\, (\vec{L} Y_{lm})^* \cdot (\vec{L} Y_{l'm'}) = \int d\Omega\, Y^*_{lm} (L^2 Y_{l'm'})$$

$$\int d\Omega\, (\hat{r} \times \vec{X}^*_{lm}) \cdot \underline{(\hat{r} \times \vec{X}_{l'm'})} = \int d\Omega\, \vec{X}^*_{lm} \cdot \vec{X}_{l'm'}$$

$$\hat{r} \cdot \left(\vec{X}^* \times (\hat{r} \times \vec{X}')\right) = \hat{r} \cdot \left(\hat{r}\,(\vec{X}^* \cdot \vec{X}') - \cancel{\vec{X}'(\hat{r} \cdot \vec{X}^*)}\right)$$

$$\underline{\int d\Omega\, \vec{X}^*_{lm} \cdot (\hat{r} \times \vec{X}_{l'm'}) = 0} \qquad \text{(HW)}.$$

$$\{a^E_{lm},\, a^M_{lm}\} \to \vec{E}, \vec{H} \qquad \Sigma.$$

$$\vec{E}, \vec{H} \to a^E_{lm},\, a^M_{lm}$$

$$\boxed{\int d\Omega\, Y^*_{lm} (\vec{r} \cdot \vec{H}) = \frac{\sqrt{l(l+1)}}{k}\, a^M_{lm}\, g_l(kr)}$$

$$\boxed{\int d\Omega\, Y^*_{lm} (\vec{r} \cdot \vec{E}) = -\frac{\sqrt{l(l+1)}}{k}\, Z_0\, a^E_{lm}\, f_l(kr)}$$

(9.123).

$r \to \text{large}$ $\quad kr \gg l$. $\quad h_l^{(1)} \to (-i)^{l+1} \dfrac{e^{ikr}}{kr}$.

outgoing waves.

$$\vec{\nabla} \times \left(h \vec{X}\right) \to ik\hat{r} \times (-i)^{l+1} \frac{e^{ikr}}{kr} = (-i)^{l} \frac{e^{ikr}}{r} \vec{X}_{lm}.$$

$$\vec{H} \to \frac{e^{ikr}}{kr} \sum_{lm} \left[a^{E}_{lm} (-i)^{l+1} \vec{X}_{lm} + a^{M}_{lm} (-i)^{l+1} \hat{r} \times \vec{X}_{lm} \right)$$

$$\vec{E} \to Z_0 \frac{e^{ikr}}{kr} \sum_{lm} \left[(-i)^{l+1} \left(a^{E}_{lm} \hat{r} \times \vec{X}_{lm} + a^{M}_{lm} \vec{X}_{lm} \right] \right.$$

$$\vec{E} = -Z_0 \hat{r} \times \vec{H} \qquad Z_0 \vec{H} = \hat{r} \times \vec{E}$$

Maxwell duality.

$$\frac{dP}{d\Omega} = r^2 \operatorname{Re}\left[\hat{r} \cdot \frac{1}{2} \vec{E} \times \vec{H}^* \right) = r^2 Z_0 \vec{H}^* \cdot \vec{H}$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| \sum_{lm} (-i)^{l+1} \left(a^{E}_{lm} \vec{X}_{lm} + a^{M}_{lm} \hat{r} \times \vec{X}_{lm} \right) \right|^2$$

$$= \frac{Z_0}{2k^2} \left| \sum_{lm} (-i)^{l+1} \left(a^{E}_{lm} \vec{X}_{lm} \times \hat{r} + a^{M}_{lm} \vec{X}_{lm} \right) \right|^2$$

(9.150)

pure mode (E or M). $\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a_{lm}|^2 \cdot |\vec{X}_{lm}|^2$

JDJ. $\vec{L} = L_x \cdot \hat{x} + L_y \cdot \hat{y} + L_z \cdot \hat{z}$

$$= \frac{1}{2}(L_+ + L_-)\hat{x} + \frac{1}{2i}(L_+ - L_-)\hat{y} + L_z \hat{z}$$

$$= \frac{1}{2} L_+ (\hat{x} + i\hat{y}) + \frac{1}{2} L_- (\hat{x} - i\hat{y}) + L_z \hat{z}$$

$$|\vec{X}_{lm}|^2 = \frac{\frac{1}{2}|L_+ Y_{lm}|^2 + \frac{1}{2}|L_- Y_{lm}|^2 + m^2 |Y_{lm}|^2}{l(l+1)}$$

$$\vec{L} = \frac{1}{i}\left(\hat{\varphi} \frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta} \frac{\partial}{\partial\phi}\right)$$ polarization direction

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$$P = \int d\Omega \, \frac{Z_0}{2k^2} \sum_{lm} \sum_{l'm'} (+i)^{l+1} (-i)^{l'+1}$$

$$\left(a^E_{lm} \vec{X}_{lm} + a^M_{lm} \hat{r} \times \vec{X}_{lm}\right)^* \cdot \left((l'm') \right)$$

cross terms vanish: $\vec{X}^* \cdot (\hat{r} \times \vec{X}')$.

squares $\rightarrow \delta_{ll'} \delta_{mm'}$.

$$\boxed{P = \frac{Z_0}{2k^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(|a^E_{lm}|^2 + |a^M_{lm}|^2 \right)} \quad (9.155)$$

5

$$\langle u \rangle = \frac{1}{4}\varepsilon_0|\vec{E}|^2 + \frac{1}{4}\mu_0|\vec{H}|^2 = \frac{1}{2}\mu_0|\vec{H}|^2$$

$$U = \int r^2 dr\, d\Omega\, u.$$

$$\frac{dU}{dr} = \int r^2 d\Omega.\ \frac{1}{2}\mu_0|\vec{H}|^2$$

$$\boxed{U' = \frac{\mu_0}{2k^2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(|a^E_{lm}|^2 + |a^M_{lm}|^2\right)} \quad (9.136)$$

$$\frac{P}{U'} = \frac{(dE/dt)}{(dE/dr)} = \text{"}v\text{"} = \frac{Z_0}{\mu_0} = \frac{1}{\mu_0}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$$

Angular momentum:

$$\vec{m} = \frac{1}{2c^2}\vec{r}\times(\vec{E}\times\vec{H}^*) = \frac{1}{2c^2}\vec{r}\times\left[\left(-\frac{Z_0}{ik}\right)(\vec{\nabla}\times\vec{H})\times\vec{H}^*\right]$$

$$\vec{\nabla}\times\vec{H} = \frac{ik}{Z_0}\vec{E}$$

$$= \frac{1}{2c^2}\left(\frac{-Z_0}{ik}\right)\left[(\vec{\nabla}\times\vec{H})\ (\vec{r}\cdot\vec{H}^*)^{\,0} - \vec{H}^*\ \vec{r}\cdot(\vec{\nabla}\times\vec{H})\right]$$

asymptotic

$$= \frac{Z_0}{2kc^2}\vec{H}^*\left[\frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{H}\right] = \frac{\mu_0}{2\omega}\vec{H}^*(\vec{L}\cdot\vec{H})$$

$$\frac{Z_0}{c} = \mu_0$$

$$\frac{d\vec{M}}{dr} = \int r^2 d\Omega \cdot \frac{\mu_0}{2\omega \cdot k^2 r^2} \sum_{l,m} \sum_{l'm'} (+i)^{l+1} (-i)^{l'+1}$$

$$\left(a^E_{lm} \vec{X}_{lm} + a^M_{lm} \hat{r} \times \vec{X}_{lm}\right)\left(a^E_{lm} \vec{L} \cdot \vec{X}_{lm} + \cdots\right)$$

Electric mode

$$\rightarrow \frac{\mu_0}{2\omega k^2} \int d\Omega \sum_{lm} \sum_{l'm'} (+i)^{l+1} (-i)^{l'+1} \left(a^E_{lm} \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}\right)$$

$$\left(a^E_{l'm'} \frac{l'(l'+1)}{\sqrt{l'(l'+1)}} Y_{l'm'}\right)$$

$M_x, M_y \rightarrow l' = l.\ m' = m \pm 1$

M_z L_z diagonal, $m' = m$ $\delta_{ll'} \delta_{mm'}$.

$$\boxed{\frac{dM_z}{dr} = \frac{\mu_0}{2\omega k^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} m \left(|a^E_{lm}|^2 + |a^M_{lm}|^2\right).}$$

Single mode.
$$\frac{dM_z/dr}{dU/dr} = \frac{M_z}{U} = \frac{m \frac{\mu_0}{2\omega k^2} |a|^2}{\frac{\mu_0}{2k^2} |a^2|}$$

$$= \frac{m}{\omega} = \frac{m\hbar}{\hbar\omega}.$$