3/12/2018 $\vec{H} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left[a_{lm}^{E} f_l(kr) \vec{X}_{lm} - \frac{i}{k} \vec{\nabla} \times \left(a_{lm}^{M} g_l(kr) \vec{X}_{lm} \right) \right]$

$$\vec{E} = Z_0 \sum_{l,m} \left[\frac{i}{k} \vec{\nabla} \times \left(a_{lm}^{E} f_l \vec{X}_{lm} \right) + a_{lm}^{M} g_l \vec{X}_{lm} \right]$$

$$\frac{dP}{d\Omega} = \frac{1}{2} \frac{|\vec{H}|^2}{Z_0} = \frac{1}{2} Z_0 |\vec{E}|^2 .$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| \sum_{l,m} (-i)^{l+1} \left(a_{lm}^{E} \vec{X}_{lm} + a_{lm}^{M} \hat{r} \times \vec{X}_{lm} \right) \right|^2$$

$$\underbrace{(-i)^{l+1} \left(a_{lm}^{E} \vec{X}_{lm} \times \hat{r} + a_{lm}^{M} \vec{X}_{lm} \right)}_{\longleftrightarrow \text{ polarization direction.}}$$

<u>Pure mode</u>. $\frac{dP_{lm}}{d\Omega} = |a_{lm}|^2 \frac{Z_0}{2k^2} |\vec{X}_{lm}|^2$

$$\boxed{P = \frac{Z_0}{2k^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(|a_{lm}^{E}|^2 + |a_{lm}^{M}|^2 \right)} \quad (9.155)$$

$\frac{dU}{dr}$, $\frac{dL_z}{dr}$.

Last project: relate to sources.

$\vec{\nabla}\cdot\vec{H} = 0$ $\quad \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \neq 0 \ldots \quad = \left(-\frac{ik}{Z_0}\vec{E}\right)$

$\vec{\nabla}\times\vec{H} = \vec{J} + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + (\vec{J}_m) - i\omega\varepsilon_0\vec{E}$

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = +i\omega\mu_0\vec{H} = ikZ_0\vec{H}$

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = +i\omega\rho$

conductor, $\vec{J} = \sigma\vec{E}$ $\quad \vec{\nabla}\times\vec{H} = \sigma\vec{E} - i\omega\varepsilon_0\vec{E}$

let $\boxed{\vec{E}' = \vec{E} + \frac{i}{\omega\varepsilon_0}\vec{J}}$ $\quad (\vec{H}' = \vec{H})$

$\vec{\nabla}\cdot\vec{E}' = \vec{\nabla}\cdot\vec{E} + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\cdot\vec{J} = \left(\frac{\rho}{\varepsilon_0}\right) + \left(\frac{i}{\omega\varepsilon_0}\right)(+i\omega\rho) = 0$

$\vec{\nabla}\cdot\vec{E}' = 0$ $\quad \vec{\nabla}\cdot\vec{H}' = 0$

$\vec{\nabla}\times\vec{E}' = \vec{\nabla}\times\vec{E} + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J} = ikZ_0\vec{H}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}$

$\vec{\nabla}\times\vec{H}' = \vec{J} - i\omega\varepsilon_0\vec{E} = -i\omega\varepsilon_0\left(\vec{E} + \frac{i}{\omega\varepsilon_0}\vec{J}\right) = \underline{\underline{\frac{-ik}{Z_0}\vec{E}'}}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}') = \vec{\nabla}(\overset{0}{\cancel{\vec{\nabla}\cdot\vec{H}'}}) - \nabla^2\vec{H}'$$

$$= \vec{\nabla}\times\left(\frac{-ik}{Z_0}\vec{E}'\right) = \frac{-ik}{Z_0}\left(ikZ_0\vec{H}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}\right)$$

$$\boxed{(\nabla^2+k^2)\vec{H}' = -\vec{\nabla}\times\vec{J}}$$

$$\left\{\begin{aligned} (\nabla^2+k^2)\vec{A} &= -\mu_0\vec{J} \\ (\nabla^2+k^2)(\vec{\nabla}\times\vec{A}) &= -\mu_0(\vec{\nabla}\times\vec{J}) \end{aligned}\right.$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}') = \vec{\nabla}(\overset{0}{\cancel{\vec{\nabla}\cdot\vec{E}'}}) - \nabla^2\vec{E}'$$

$$= \vec{\nabla}\times\left(ik\,Z_0\vec{H}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}\right)$$

$$= ikZ_0\left(\frac{-ik}{Z_0}\vec{E}'\right) + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times(\vec{\nabla}\times\vec{J})$$

$$\boxed{(\nabla^2+k^2)\vec{E}' = -\frac{Z_0}{k}\vec{\nabla}\times(\vec{\nabla}\times\vec{J})}$$

Recall: $\nabla^2(\vec{r}\cdot\vec{v}) = 2(\vec{\nabla}\cdot\vec{v}) + \vec{r}\cdot(\nabla^2\vec{v})$

$$(\nabla^2+k^2)(\vec{r}\cdot\vec{H}') = -\vec{r}\cdot(\vec{\nabla}\times\vec{J}) = -i\vec{L}\cdot\vec{J}$$

$$(\nabla^2+k^2)(\vec{r}\cdot\vec{E}') = +\frac{Z_0}{k}\vec{L}\cdot(\vec{\nabla}\times\vec{J})$$

Green's Function $\quad G = \sum_{lm} ik\, j_l h_l\, Y^* Y.$

$$(\vec{r}\cdot\vec{E}') = -\int d^3x' \sum_{l=0}^{\infty}\sum_{m=-l}^{l} ik\, j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y^*_{lm}(\theta',\phi')\, Y_{lm} \times \frac{Z_0}{k}\, \vec{L}\cdot(\vec{\nabla}\times\vec{J})$$

(f).

$r\to\infty$, $r_> = r$, $r_< = r'$

recall. (9.123)

$$\frac{-k}{\sqrt{l(l+1)}}\int d\Omega\, Y^*_{lm}(\theta,\phi)\,(\vec{r}\cdot\vec{E}) = Z_0\, a^E_{lm}\, h_l^{(1)}(kr)$$

$$\boxed{a^E_{lm} = \frac{ik}{\sqrt{l(l+1)}}\int d^3x'\, j_l(kr')\, Y^*_{lm}(\theta',\phi')\, \vec{L}'\cdot(\vec{\nabla}'\times\vec{J})} \quad (9.165)$$

$k = \frac{k^3}{k^2}$

$$(\vec{r}\cdot\vec{H}') = +\int d^3x' \sum_{lm} ik\, j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \times i\vec{L}'\cdot\vec{J}$$

(g).

$$\frac{k}{\sqrt{l(l+1)}}\int d\Omega\, Y^*_{lm}(\theta,\phi)\,(\vec{r}\cdot\vec{H}) = a^M_{lm}\, h_l^{(1)}(kr)$$

$$\boxed{a^M_{lm} = \frac{-k^2}{\sqrt{l(l+1)}}\int d^3x'\, j_l(kr')\, Y^*_{lm}(\theta',\phi')\, \vec{L}'\cdot\vec{J}} \quad (9.165)$$

Familiar (?) forms. undo $\vec{L}$.

$$\vec{L}\cdot\vec{J} = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{J} = \frac{1}{i}\,\vec{r}\cdot(\vec{\nabla}\times\vec{J})$$

$$\vec{L}\cdot(\vec{\nabla}\times\vec{J}) = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot(\vec{\nabla}\times\vec{J}) = \frac{1}{i}\,\vec{r}\cdot\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{J})\right)$$

$$= \frac{1}{i}\,\vec{r}\cdot\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \nabla^2\vec{J}\right)$$

$$= \frac{1}{i}\left[r\frac{\partial}{\partial r}(\vec{\nabla}\cdot\vec{J}) - \vec{r}\cdot\nabla^2\vec{J}\right)$$

$$= \frac{1}{i}\left[r\frac{\partial}{\partial r}(\vec{\nabla}\cdot\vec{J}) - \left(\nabla^2(\vec{r}\cdot\vec{J}) - 2\,\vec{\nabla}\cdot\vec{J}\right)\right]$$

$$= i\,\nabla^2(\vec{r}\cdot\vec{J}) - ir\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)(\vec{\nabla}\cdot\vec{J})$$

$\uparrow$ $i\omega\rho = ick\rho$.

$$a^E_{\ell m} = \frac{ik}{\sqrt{\ell(\ell+1)}}\int d^3x'\, j_\ell(kr')\,Y^*_{\ell m}(\theta',\phi')\left[\frac{r'}{r'^2}\frac{\partial}{\partial r'}(r'^2 c\rho) - \nabla^2(\vec{r}'\cdot\vec{J})\right]$$

parts (twice) $\rightarrow k^2$

$$a^E_{\ell m} = \frac{-ik^2}{\sqrt{\ell(\ell+1)}}\int d^3x'\, Y^*_{\ell m}(\theta',\phi')\left[c\rho\,\frac{\partial}{\partial r'}(r' j_\ell(kr')) + ik(\vec{r}'\cdot\vec{J})\,j_\ell(kr')\right]$$

9.167

6

$$a^M_{lm} = \frac{-ik^2}{\sqrt{l(l+1)}} \int d^3x' \, Y^*_{lm}(\theta',\phi') \, j_l(kr') \, \vec{\nabla}' \cdot (\vec{r}' \times \vec{J})$$

9.167

exact ↑

long wavelength ↓.

$kd \ll 1$ $\quad j_l \to \frac{(kr')^l}{(2l+1)!!}$ $\quad k(\vec{r}, \vec{J}) \ll c\rho$

$$a^E_{lm} \to \frac{-ik^2}{\sqrt{l(l+1)}} \int d^3x' \cdot Y^*_{lm}(\theta',\phi') \, c\rho \, \frac{(l+1)(kr')^l}{(2l+1)!!}$$

$$a^E_{lm} = \frac{-ick^{l+2}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} \int d^3x' \, Y^*_{lm}(\theta',\phi') \, r'^l \rho$$

9.169

$Q_{lm} = q_{lm}$

$$a^m_{lm} = \frac{ik^{l+2}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} \cdot \sqrt{\frac{l+1}{l}} \left(-\frac{1}{l+1} \int d^3x' \, Y^*_{lm} \, r'^l \, \vec{\nabla}' \cdot (\vec{r}' \times \vec{J}) \right).$$

M_{lm}. 9.171