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$$\vec{E}_{sc} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left(\frac{1}{2} \alpha_l^{\pm} h_l^{(1)}(kr) \vec{X}_{l,\pm 1} \pm \frac{i}{2} \beta_l \frac{1}{k} \vec{\nabla} \times \left(h \vec{X} \right) \right)$$

$$c\vec{B}_{sc} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[-i \frac{1}{2} \alpha_l \vec{\nabla} \times \left(h \vec{X} \right) \mp i \frac{1}{2} \beta_l h_l^{(1)}(kr) \vec{X}_{l,\pm 1} \right]$$

(10.57).

$$P_{scatt} = -\frac{a^2}{2\mu_0} \int_{r=a} d\Omega \; \mathrm{Re} \left[\vec{E}_{sc} \cdot \left(\hat{r} \times \vec{B}^{*}_{sc} \right) \right] \quad (10.58)$$

$$P_{abs} = +\frac{a^2}{2\mu_0} \int_{r=a} d\Omega \; \mathrm{Re} \left[\vec{E} \cdot \left(\hat{r} \times \vec{B}^{*} \right) \right] \quad (10.59)$$

$$\nabla \times \left(f \vec{X} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left(r f(r) \right) \hat{r} \times \vec{X} + i \sqrt{l(l+1)} \frac{f}{r} Y_{lm} \hat{r}$$

$$\vec{E}_{sc} \cdot \left(\hat{r} \times \vec{B}^{*}_{sc} \right) = \sum_{l l'} i^l (-i)^{l'} \sqrt{4\pi(2l+1)} \sqrt{4\pi(2l'+1)}$$

$$\left[\frac{1}{2} \alpha_l h_l \vec{X}_l \pm \frac{i}{k} \frac{1}{2} \beta_l \left(\hat{r} \times \vec{X} \frac{1}{r} \frac{\partial}{\partial r}(rh) + \hat{r} \ldots \right) \right]$$

$$\cdot \left[\hat{r} \times \left(\frac{-i}{k} \frac{1}{2} \alpha_{l'} \left(\hat{r} \times \vec{X}_{l'} \frac{1}{r} \frac{\partial}{\partial r}(rh) + \hat{r}(\ldots) \right) \right) \mp i \frac{1}{2} \beta_{l'} h_{l'} \vec{X}_{l'} \right]^{*}$$

$$\rightarrow \frac{1}{2} \alpha_l \frac{1}{2} \alpha_{l'}^{*} \frac{i}{k} \vec{X}_l \cdot \vec{X}^{*}_{l'} \, h_l \frac{1}{r} \frac{\partial}{\partial r} \left(r h^{*}_{l'} \right) + \frac{1}{2} \beta_l \frac{1}{2} \beta^{*}_{l'} \frac{i}{k} h_l \frac{1}{r} \ldots \quad \vec{X}_l \cdot \vec{X}^{*}_{l'}$$

$$h^{(1)} h^{(2)\prime} - h^{(2)} h^{(1)\prime} = (j + i n)(j' - i n') - (j - i n)(j' + i n') = -2i (j n' - n j') = -2i \left(\frac{1}{x^2} \right)$$

$$= \ldots \frac{1}{x^2}$$

$$\sigma_{scatt} = \frac{\pi}{2k^2} \sum_{\ell=1}^{\infty} (2\ell+1)\left(|\alpha_\ell^{\pm}|^2 + |\beta_\ell^{\pm}|^2\right)$$ (10.61)

$$\sigma_{abs} = \frac{\pi}{2k^2} \sum_{\ell=1}^{\infty} (2\ell+1)\left[2 - |1+\alpha_\ell^{\pm}|^2 - |1+\beta_\ell^{\pm}|^2\right]$$

$r \to \infty$ $\quad h_\ell^{(1)} \to (-i)^{\ell+1} \frac{e^{ikr}}{r}$

$$\frac{d\sigma}{d\Omega} = \frac{r^2 \cdot \frac{1}{2Z_0}|\vec{E}_{sc}|^2}{\frac{1}{2Z_0}|E_0|^2} = \frac{r^2 |\vec{E}_{sc}|^2}{2}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{scatt} = \frac{\pi}{2k^2} \left| \sum_{\ell=1}^{\infty} \sqrt{2\ell+1} \left(\alpha_\ell^{\pm} \vec{X}_{\ell,\pm 1} \pm \beta_\ell \, \hat{r} \times \vec{X}_{\ell,\pm 1}\right)\right|^2$$

$\rightarrow \sigma_{scatt}$

"Optical Theorem".

$\sigma_{tot} = \sigma_{abs} + \sigma_{scatt}$

$$= \frac{\pi}{2k^2} \sum_{l=1}^{\infty} (2l+1) \Big[2 - (1+\alpha_l^*)(1+\alpha_l) - (1+\beta_l^*)(1+\beta_l) + |\alpha_l|^2 + |\beta_l|^2 \Big]$$

$$\cancel{2} - \cancel{1} - \alpha - \alpha^* - \cancel{|\alpha|^2} - \cancel{1} - \beta - \beta^* - \cancel{|\beta|^2} + \cancel{|\alpha|^2} + \cancel{|\beta|^2}$$

$$= -(\alpha + \alpha^* + \beta + \beta^*) = -2\,\mathrm{Re}(\alpha + \beta).$$

$$\boxed{\sigma = \frac{-\pi}{2k^2} \sum_{l=1}^{\infty} (2l+1)\, \mathrm{Re}(\alpha_l^{\pm} + \beta_l^{\pm})}$$

Linear in $(\alpha, \beta) \to$ amplitude.

Forward "scattering amplitude" $\vec{E}_{sc} = \vec{F} \cdot \frac{e^{ikr}}{r}.$

$m = \pm 1$. $Y_{l,1} \propto \sin\theta$ $\quad L_z Y_{l,1} \propto \sin\theta \to 0$ as $\theta \to 0$.

$$\vec{X}_{l,\pm 1} = \frac{\vec{L} Y_{l,\pm 1}}{\sqrt{l(l+1)}} = \frac{\left[\frac{1}{2}(L_+ + L_-)\hat{x} + \frac{1}{2i}(L_+ - L_-)\hat{y} + \hat{z} L_z\right] Y}{\sqrt{l(l+1)}}.$$

$$L_{\mp} Y_{\ell \pm 1} \sim Y_{\ell 0} \sim P_\ell(\cos\theta) \to 1.$$

$$\vec{X}_{\ell,1}(\theta=0) = \frac{\frac{1}{2}(\hat{x}+i\hat{y}) L_- Y_{1,1}}{\sqrt{\ell(\ell+1)}} = \frac{1}{2}(\hat{x}+i\hat{y}) \frac{\cancel{\sqrt{(\ell+m)(\ell-m+1)}}}{\cancel{\sqrt{\ell(\ell+1)}}} Y_{\ell,0}$$

$$= \frac{1}{2}(\hat{x}+i\hat{y}) \sqrt{\frac{2\ell+1}{4\pi}} (1)$$

$$\vec{X}_{\ell,-1} = \frac{\frac{1}{2}(\hat{x}-i\hat{y}) L_+ Y_{1,-1}}{\sqrt{\ell(\ell+1)}} = \frac{1}{2}(\hat{x}-i\hat{y}) \sqrt{\frac{2\ell+1}{4\pi}}$$

$$\vec{E}_{sc} = \frac{e^{ikr}}{r} \vec{F}_{sc} = \sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} (-i)^{\ell+1} \frac{e^{ikr}}{kr}$$

$$\times \left(\frac{1}{2} \alpha_\ell \vec{X}_{\ell,\pm 1} \pm i \beta_\ell^{\pm} \hat{r} \times \vec{X}_{\ell,\pm 1} \right)$$

$$\vec{F}_{sc} = \frac{-i}{2k} \sum_{\ell=1}^{\infty} (2\ell+1) \cdot \frac{1}{2} \cdot \left(\alpha_\ell (\hat{x} \pm i\hat{y}) \pm i\beta_\ell \, \hat{z} \times (\hat{x} \pm i\hat{y}) \right)$$

$$\hat{z} \times (\hat{x} \pm i\hat{y}) = \hat{y} \mp i\hat{x} = \mp i(\hat{x} \pm i\hat{y})$$

$$\vec{F}_{sc} = \frac{-i}{4k} (\hat{x} \pm i\hat{y}) \sum_{\ell=1}^{\infty} (2\ell+1)(\alpha_\ell^{\pm} + \beta_\ell^{\pm})$$

Normalized :

$$\vec{\varepsilon}_{\pm}^{\,*} \cdot \vec{f}_{sc} = \frac{(\vec{\varepsilon}_{\pm})^{*} \cdot \vec{F}_{sc}}{E_0} = \frac{\frac{(\hat{x} \pm i\hat{y})}{\sqrt{2}} \cdot \vec{F}_{sc}}{\sqrt{2}} = \frac{1}{2}(\hat{x} \pm i\hat{y})^{*} \cdot \vec{F}$$

$$= \frac{-i}{4k} \sum_{l=1}^{\infty} (2l+1)(\alpha_l^{\pm} + \beta_l^{\pm}).$$

$$\sigma_{tot} = -\frac{\pi}{k^2} \sum_{l} (2l+1)\, \mathrm{Re}(\alpha_l + \beta_l).$$

$$\boxed{\sigma_{tot} = \frac{4\pi}{k} \,\mathrm{Im}\left[\vec{\varepsilon}_0^{\,*} \cdot \vec{f}(\theta=0)\right]} \quad (10.139).$$

"Optical Theorem".

$$\vec{E}_0 = (\hat{x} \pm i\hat{y}) e^{ikz}$$

$$= \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[j_l(kr) \vec{X}_{l,\pm 1} \pm \frac{1}{k} \vec{\nabla} \times \left(j_l \vec{X}_{l,\pm 1} \right) \right]$$

$$\vec{E}_{sc} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[\frac{1}{2} \alpha_l h_l^{(1)} \vec{X}_{l,\pm 1} \pm \frac{1}{2} \beta_l \frac{1}{k} \vec{\nabla} \times \left(h_l \vec{X} \right) \right]$$

$(\alpha_l, \beta_l) \Leftrightarrow$ Boundary conditions

"conducting" $\rightarrow$ $\vec{E}_{\parallel}\Big|_S = 0$ $\quad \vec{H}_{\perp}\Big|_S = 0$.

$$\Rightarrow \quad \frac{1}{2}\alpha_l h^{(1)} + j_l = \frac{1}{2}\alpha_l h^{(1)} + \frac{1}{2}\left(h^{(1)} + h^{(2)}\right) = 0$$

(Almost) as easy: $\vec{E}_{\parallel}\Big|_S = Z_S \, \hat{r} \times \vec{H}_{\parallel}\Big|_S$

Z_S : "surface impedance"

$$\vec{\nabla} \times (f\vec{X}) = \frac{1}{r}\frac{\partial}{\partial r}(rf)\,\hat{r} \times \vec{X} + i\sqrt{l(l+1)}\,\frac{f}{r}\,Y_{lm}\,\hat{r}$$

$c\beta = Z_0 H$

α-term:

$$\frac{1}{2}\alpha_\ell h^{(1)} + \frac{1}{2}\left(h^{(1)} + h^{(2)}\right)$$

$$= i\frac{Z_s}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left[x\left(\frac{1}{2}\alpha h^{(1)} + \frac{1}{2}h^{(1)} + \frac{1}{2}h^{(2)}\right)\right]\Bigg|_{x=ka}$$

$$(1+\alpha_\ell)\left[h_\ell^{(1)} - \frac{iZ_s}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left[x\,h^{(1)}(x)\right]\right]$$

$$= -h^{(2)} + \frac{iZ_s}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left(x\,h^{(2)}\right).$$

$$\boxed{1+\alpha_\ell = -\frac{h_\ell^{(2)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left(x\,h^{(2)}\right)}{h_\ell^{(1)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left(x\,h^{(1)}\right)}}\quad x = ka$$

β_ℓ duality. $\vec{E} \to c\vec{B} = Z_0\vec{H}$

$c\vec{B} = Z_0\vec{H} \to -\vec{E}$.

$$\frac{Z_s}{Z_0} \to \frac{Z_0}{Z_s}$$

$$\hat{r}\times\vec{E}_{\parallel}\Big|_s = Z_s\left(\hat{r}\times(\hat{r}\times\vec{H}_{\parallel})\right)$$

$$= Z_s\left(\hat{r}(\hat{r}\cdot\vec{H}_{\parallel}) - \vec{H}_{\parallel}(\hat{r}\cdot\hat{r})\right) = Z_s(\vec{H}_{\parallel})$$

$$\boxed{\vec{H}_{\parallel} = -\frac{1}{Z_s}\hat{r}\times\vec{E}_{\parallel}}$$