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$$\vec{E}_{sc} = \sum_{\ell m}^{\infty} i^{\ell} \sqrt{4\pi(2\ell+1)} \left[\frac{1}{2}\alpha_{\ell}\, h_{\ell} \vec{X}_{\ell} \pm \frac{i}{2}\beta_{\ell} \frac{1}{k} \vec{\nabla}\times\left(h\vec{X}\right) \right]$$

$$c\vec{B}_{sc} = \sum_{\ell m}^{\infty} i^{\ell} \sqrt{4\pi(2\ell+1)} \left[-\frac{i}{k}\frac{1}{2}\alpha_{\ell} \vec{\nabla}\times\left(h\vec{X}\right) \mp i\frac{1}{2}\beta_{\ell}\, h\vec{X} \right]$$

$\vec{E}_0$ $\quad \frac{\alpha}{2} \to 1$ $\quad \frac{\beta}{2} \to 1$ $\quad h \to j$

Boundary condition: $\boxed{\vec{E}_{\parallel}\big|_S = Z_S\, \hat{r}\times\vec{H}_{\parallel}\big|_S}$

$$\frac{1}{2}\alpha_{\ell} h_{\ell}^{(1)} + \left(\frac{1}{2}h^{(1)} + \frac{1}{2}h^{(2)}\right) = \frac{iZ_S}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left[x\left(\frac{1}{2}\alpha h^{(1)} + \frac{1}{2}h^{(1)} + \frac{1}{2}h^{(2)}\right)\right]_{x=ka}$$

$$(1+\alpha_{\ell})\left[h_{\ell}^{(1)} - i\frac{Z_S}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left(x h^{(1)}\right)\right] = -h^{(2)} + i\frac{Z_S}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left(x h^{(2)}\right)$$

$$1+\alpha_{\ell} = -\frac{h_{\ell}^{(2)} - \frac{iZ_S}{Z_0}\frac{1}{x}\partial_x\left(x h^{(2)}\right)}{h_{\ell}^{(1)} - i\frac{Z_S}{Z_0}\frac{1}{x}\partial_x\left(x h^{(1)}\right)}$$

$$\hat{r}\times\vec{E}_{\parallel}\big|_S = Z_S\left(\hat{r}\times\left(\hat{r}\times\vec{H}_{\parallel}\right)\right) = -\vec{H}_{\parallel}\big|_S.$$

$$\vec{H}_{\parallel} = -\frac{1}{Z_S}\,\hat{r}\times\vec{E}_{\parallel}$$

$\beta_{\ell}:\ \frac{Z_S}{Z_0} \to \frac{Z_0}{Z_S}$

(2)

conducting : $\vec{E}_{\parallel}\big|_s = 0$ $\quad (Z_s = 0)$

$$1 + \alpha_\ell = -\frac{h^{(2)}}{h^{(1)}} = -\frac{(j - in)}{(j + in)} = \frac{n^2 - j^2 + 2ijn}{j^2 + n^2}$$

$$1 + \alpha_\ell = -\frac{(j - in)}{(j + in)} = \frac{in - j}{in + j} = \frac{i(n + ij)}{i(n - ij)} = \frac{n + ij}{n - ij}$$

$$= \frac{(R e^{i\delta})}{(R e^{-i\delta})} = e^{2i\delta_\ell}$$

pure imaginary. $Z_s \to 0$ $Z_s \to \infty$.

$$e^{2i\delta_\ell} = \cos 2\delta_\ell + i \sin 2\delta_\ell = \frac{n^2 - j^2 + 2inj}{j^2 + n^2}.$$

$$\cos\delta_\ell = \frac{n}{\sqrt{n^2 + j^2}} \qquad \sin\delta_\ell = \frac{j}{\sqrt{n^2 + j^2}} \qquad \tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}$$

$$1 + \beta_\ell = -\frac{\frac{1}{x}\frac{d}{dx}(x h^{(2)})}{\frac{1}{x}\frac{d}{dx}(x h^{(1)})} = e^{2i\delta'_\ell}$$

$$\tan\delta'_\ell = \frac{\frac{1}{x}\frac{d}{dx}(x\, j_\ell(x))}{\frac{1}{x}\frac{d}{dx}(x\, n_\ell(x))}\Bigg|_{x = ka}$$

$ka \ll 1$. $\tan\delta_\ell = \dfrac{(ka)^\ell/(2\ell+1)!!}{-(ka)^{-\ell-1}\cdot(2\ell-1)!!} = -\dfrac{(ka)^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!}$

$$\tan\delta_\ell' = \frac{\ell+1}{\ell}\cdot\frac{(ka)^{2\ell+1}}{(2\ell+2)!!\,(2\ell-1)!!}$$

$$\alpha_\ell = e^{2i\delta_\ell}-1 \simeq 2i\delta_\ell. \qquad \beta_\ell \simeq 2i\delta_\ell'$$

$\ell=1$ $\delta_1 = -\dfrac{(ka)^3}{3}$ $\delta_1' = +\dfrac{2}{3}(ka)^3$

$$\alpha_1 = -\frac{2i}{3}(ka)^3 \qquad \beta_1 = +\frac{4i}{3}(ka)^3$$

$$\frac{d\sigma}{d\Omega} \simeq \frac{\pi}{2k^2}\cdot(3)^{2(\ell+1)}\left|-\frac{2}{3}(ka)^3\vec{X}_1 \pm i\left[\frac{4}{3}(ka)^3\hat{r}\times\vec{X}_1\right]\right|^2$$

$$= \frac{\pi}{2k^2}\cdot 3\cdot\frac{4}{9}(ka)^6\left|\vec{X}_{1,\pm1} \mp 2i\,\hat{r}\times\vec{X}_{1,\pm1}\right|^2$$

$$\vec{X}_{1,1} = \sqrt{\frac{3}{16\pi}}\left(\hat{\theta}+i\cos\theta\,\hat{\phi}\right)e^{i\phi}$$

$$\hat{r}\times\vec{X} = \sqrt{\frac{3}{16\pi}}\left(-i\cos\theta \quad +\hat{\phi}\right)e^{i\phi}$$

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{2k^2} \cdot \frac{\cancel{8} \cdot 4}{\cancel{9}} (ka)^6 \cdot \frac{\cancel{3}}{16\pi} \left| \hat{\theta}(1 - 2\cos\theta) + \hat{\phi}(i\cos\theta - 2i) \right|^2$$

$$(1-2\cos\theta)^2 + (\cos\theta - 2)^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6 \left(\frac{5}{8}(1+\cos^2\theta) - \cos\theta \right)}$$

$$\alpha_\ell = e^{2i\delta_\ell} - 1 = \cos 2\delta_\ell - 1 + i \sin 2\delta_\ell$$

$$= i \cdot 2\sin\delta_\ell \cos\delta_\ell - 2 \cdot \sin^2\delta_\ell$$

$$= 2i \sin\delta_\ell (\cos\delta_\ell + i\sin\delta_\ell) = 2i\,\delta_\ell\, e^{i\delta_\ell}.$$

$$\sigma_{scatt} = \frac{\pi}{2k^2} \cdot \sum_{\ell=1}^{\infty} (2\ell+1)\left(|\alpha_\ell|^2 + |\beta_\ell|^2 \right).$$

$$\boxed{\sigma_{scatt} = \frac{2\pi}{k^2} \sum_{\ell=1}^{\infty} (2\ell+1)\left[\sin^2\delta_\ell + \sin^2\delta_\ell' \right]}$$

Not in JDJ, but should be.

(5)

$ka \ll 1$ $\quad \dfrac{(ka)^{2l+1}}{(2l+1)!!\,(2l-1)!!} \rightarrow$ rapidly falling

$\rightarrow$ dipole.

$ka \gg 1$. $\quad l < ka. \rightarrow$ small argument. $\rightarrow$

$l \lesssim ka$ $\quad \tan\delta_l = \dfrac{j_l}{n_l} \sim \dfrac{\cancel{\frac{1}{x}}\sin(x - \frac{l\pi}{2})}{-\cancel{\frac{1}{x}}\cos(x - \frac{l\pi}{2})}$

$$= \tan\left(\frac{l\pi}{2} + \pi - x\right)$$

$$\delta_l = \pi\left(1 + \frac{l}{2}\right) - x \quad \rightarrow \text{random-ish.}$$

$$\sigma = \frac{2\pi}{k^2} \sum_{l=1}^{\infty} (2l+1)\cdot\left(\sin^2\delta_l + \sin^2\delta_l'\right)$$

$$\sim \frac{2\pi}{k^2} \sum_{l=1}^{(ka)} (2l+1)\left(\frac{1}{2} + \frac{1}{2}\right)$$

$\hookrightarrow 1,\ 1+3=4,\ 1+3+5=9 \rightarrow (ka)^2$

$$\sigma \sim \frac{2\pi}{k^2}\cdot(ka)^2 = 2\pi a^2$$