3/26/2018 **Diffraction** §10.5 (aperture/edge).

short wavelength. $kd \gg 1$

Start with scalar: $(\nabla^2 + k^2)\psi = 0$ $(\psi = \Phi, \vec{A}, \vec{E}, \vec{B})$

Recall: Green's Theorem (1.35)

$$\int_V d^3x' \left[\phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right] = \oint_S da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

let: $\phi = G$ $(\nabla^2 + k^2) G = -\delta(\vec{x} - \vec{x}')$ $G = \frac{e^{ik|\vec{x} - \vec{x}'|}}{4\pi |\vec{x} - \vec{x}'|}$

$$\int d^3x' \left[G(-k^2\psi) - \psi\left(-k^2 - \delta(\vec{x} - \vec{x}')\right) \right] = \oint da' \left(G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right)$$

$$\psi(\vec{x}) = \oint_S d^2a' \left(G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right)$$

Two kinds of problems

Ⓐ

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$S = S_1$ ["sources"] ⊕ $S_2 = \{ r' \to \infty \}$.

On S_2: $\psi \to \frac{e^{ikr'}}{r'} f(\theta', \phi')$ $\quad G \to \frac{1}{4\pi} \frac{e^{ikr'}}{r'} e^{-ik\hat{r}\cdot\vec{x}}$

$$\frac{\partial \psi}{\partial n'} = \hat{r}' \cdot \vec{\nabla}' \psi = ik\psi + O\left(\frac{\psi}{r'}\right)$$

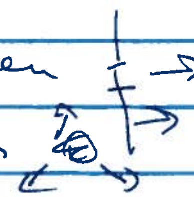
$$\frac{\partial G}{\partial n'} = \hat{r}' \cdot \vec{\nabla}' G = ikG + O\left(\frac{G}{r'}\right)$$

$$\int_{S_2} d^2a' \left(G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right) = \int_{r'\to\infty} r'^2 d\Omega' \left(\frac{1}{4\pi} \frac{e^{ikr'}}{r'} e^{-ik\hat{r}'\cdot\vec{x}}\, ik \frac{e^{ikr'}}{r'} f - \frac{ik}{4\pi} \frac{e^{ikr'}}{r'} e^{-ik\hat{r}'\cdot\vec{x}} \frac{e^{ikr'}}{r'} f \right) + O\left(\frac{1}{r'}\right) \to 0.$$

So: only S_1.

Next

* reverse $\hat{n}'$ (into V → out of screen or scatterer)



* let $\vec{R} = \vec{x} - \vec{x}'$

$$\vec{\nabla}' R = -\vec{\nabla} R = -\hat{R}$$

$$\psi(\vec{x}) = -\frac{1}{4\pi} \int_{S_1} d^2a' \frac{e^{ikR}}{R} \left(\hat{n}' \cdot \vec{\nabla}' \psi + ik\, \hat{n}' \cdot \hat{R}\, \psi \left(1 + \frac{i}{kR}\right) \right)$$

Kirchhoff integral formula. (10.79)

Apply · ($\frac{1}{kR} \to 0$) ($r \to$ large)

$\psi_0 = e^{i\vec{k}_0\cdot\vec{x}'}$

$$G = \frac{e^{ikR}}{4\pi R} \to \frac{1}{4\pi}\frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'}$$

S = {screen with openings}

Approximate : $\psi = 0$ on "screen", $\psi = \psi_0$ "openings"

$$\psi = -\frac{1}{4\pi}\int_{\text{openings}} d^2a'\, \hat{n}\cdot\left(i\vec{k}_0 + ik\hat{R}\right) e^{i\vec{k}_0\cdot\vec{x}'} \frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'}$$

$\vec{k}_{in} = \vec{k}_0$ $\quad$ $\vec{k}_{out} = k\hat{R}$

$$\psi = -\frac{ik}{4\pi}\frac{e^{ikr}}{r}\int_{\text{openings}} d^2a' \left(\cos\theta_{in} + \cos\theta_{out}\right) e^{i(\vec{k}_{in} - \vec{k}_{out})\cdot\vec{x}'}$$

Objection! Both ψ, $\frac{\partial\psi}{\partial n}$ specified.

Fix: $G_N, G_D = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} \pm \frac{e^{ikR_{image}}}{R_{image}}\right)$

$G_D\big|_S$ or $\frac{\partial G_N}{\partial n}\big|_S$ vanish $\longleftrightarrow$ other term doubles

Dirichlet (10.85)

$\theta_{in} = \theta_{out} + O\left(\frac{1}{ka}\right)$..

Why is this anything to do?

Look at: $\frac{dy}{dx} = x \qquad y(0) = 1$

$$\int_0^x \left(\frac{dy}{dx}\right) dx = y(x) - y(0) = \int_0^x y(x')\,dx'$$

$$y(x) = 1 + \int_0^x y(x')\,dx'$$

Differential Eq. + BC → Integral Eq.

Let. $y_0(x) = 0$ Iterate $y_{n+1}(x) = \int_0^x y_n(x)\,dx$

$$y_1 = 1 + \int_0^x (0)\,dx' = 1.$$

$$y_2 = 1 + \int_0^x (1)\,dx' = 1 + x$$

$$y_3 = 1 + \int_0^x (1 + x')\,dx' = 1 + x + \tfrac{1}{2}x^2.$$

$$y_n = \sum_{k=0}^{n-1} \frac{1}{k!} x^k \to e^x. \quad ✓$$

"Contraction Map", $x \to f(x)$.

$$d(fx, fy) \le \lambda\, d(x,y)$$
$$0 \le \lambda < 1$$

converges to unique fixed point.

$$d(f^n x, f^n y) \le \lambda^n d(x,y) \to 0.$$

Example. opening. $\left(-\frac{a}{2} < x < \frac{a}{2}\right)$, $\left(-\frac{b}{2} < y < \frac{b}{2}\right)$

(b → large)

$$\psi = \frac{-ik}{2\pi} \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy' \cdot (2)\, e^{-ik(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}$$

$$= \frac{-ik}{2\pi} \int_{-a/2}^{a/2} dx'\, e^{-ik\sin\theta\cos\phi\, x'} \int_{-b/2}^{b/2} dy'\, e^{-ik\sin\theta\sin\phi\, y'}$$

$\simeq \delta(k\sin\theta\sin\phi)$.

$\to (b)\cdot(\phi = 0)$.

$$= \frac{-ikb}{2\pi} \int_{-a/2}^{a/2} dx'\, e^{-ik\sin\theta\, x'}$$

$$\hookrightarrow \left.\frac{e^{-ikx'\sin\theta}}{-ik\sin\theta}\right|_{-a/2}^{a/2}$$

$$= \frac{-ikb}{2\pi} \cdot \frac{a}{\frac{1}{2}ka\sin\theta} \left(\frac{e^{i\frac{ka}{2}\sin\theta} - e^{-i\frac{ka}{2}\sin\theta}}{2i} \right)$$

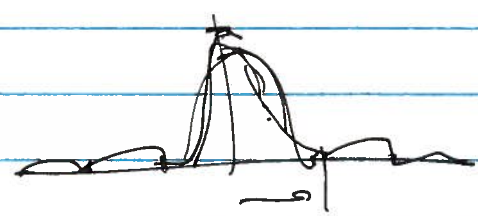
$$\boxed{\psi = \frac{-ikab}{2\pi} \cdot \frac{\sin\left(\frac{1}{2}ka\sin\theta\right)}{\left(\frac{1}{2}ka\sin\theta\right)}}$$

$$\vec{S} = \hat{k}\,|\psi|^2$$

$$\vec{S}_{in} \cdot \hat{k} = |\psi_0|^2 = 1$$

$$r^2\,\vec{S}_{out} \cdot \hat{r} = \frac{k^2 a^2 b^2}{4\pi^2} \cdot \frac{\sin^2(\frac{1}{2} ka \sin\theta)}{(\frac{1}{2} ka \sin\theta)^2}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{k^2 a^2 b^2}{4\pi^2} \frac{\sin^2(\frac{1}{2} ka \sin\theta)}{(\frac{1}{2} ka \sin\theta)^2}}$$



$$\frac{1}{2} ka \sin\theta = n\pi = \frac{1}{2} \cdot \frac{2\pi}{\lambda} a \sin\theta$$

$$\boxed{n\lambda = a \sin\theta}$$

$$\frac{1}{\pi} \int_{-n\pi}^{n\pi} \frac{\sin^2 x}{x^2}\, dx =$$

0.902823... $n=1$

0.949939... $n=2$

0.966410... $n=3$

$$\simeq \boxed{1 - \frac{1}{\pi n^2}}$$

~~$a = b$~~ ~~$T = \frac{(ka)^2}{4\pi}$~~

$$T = \frac{kab}{4\pi}$$