3/30/2018 Chapter 11 $\boxed{\text{Relativity}}$

We have seen:

maxwell $\rightarrow$ $\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = (-\rho/\epsilon_0)$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = (-\mu_0\vec{J}).$$

$\leftrightarrow$ wave @ speed $v = c$ w/r to ?? as seen by ??.

Newtonian physics $\leftrightarrow$ "Galilean Relativity"

$$\vec{x} \rightarrow \vec{x}' = \vec{x} + \vec{v}t \qquad (\vec{v} = \text{constant})$$

$$t' = t$$

$$\boxed{m_i \frac{d^2\vec{x}_i'}{dt^2} = -\frac{\partial}{\partial x_i'}\sum_{j\neq i} V(|\vec{x}_i' - \vec{x}_j'|)}$$ unchanged.

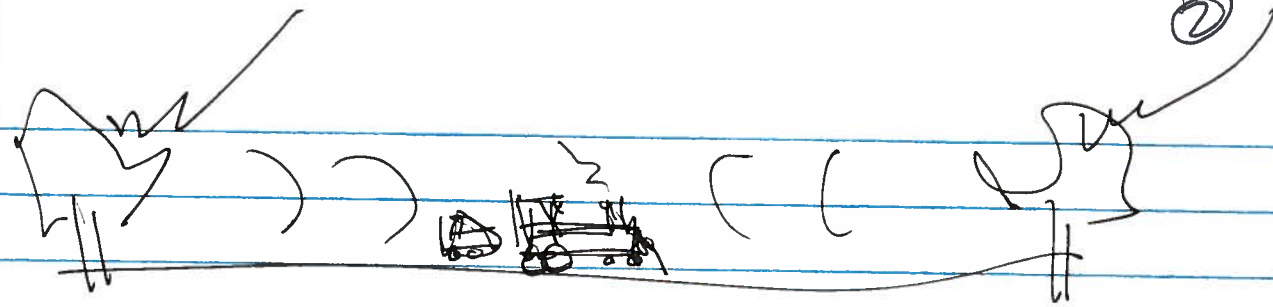
$\Phi(\vec{x} + \vec{v}t, t)$ $\qquad \frac{\partial}{\partial t'} \rightarrow \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$

$$\nabla^2\Phi - \frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2} + 2\vec{v}\cdot\vec{\nabla}\frac{\partial\Phi}{\partial t} + (\vec{v}\cdot\vec{\nabla})^2\Phi\right)$$

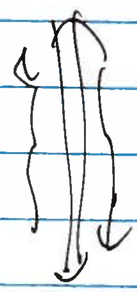
??

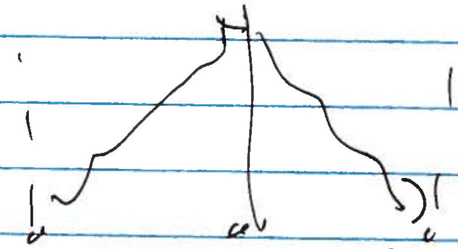
~~Restrict to special observer (Ether at rest)~~

$\boxed{\text{Different transformation}}$



Simultaneous on ground ⇒ not on train.
time depends on observer!



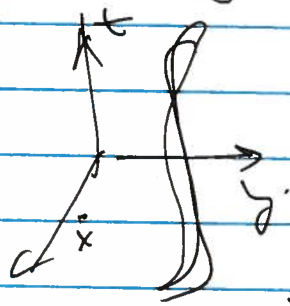


$$\Delta t_0 = \frac{2L}{c}$$

$$c^2 \cdot \Delta t_1^2 = v^2 \Delta t_1^2 + L^2$$

$$\Delta t' = 2\Delta t_1 = \frac{2L}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

Machinery:



Trajectory: $\vec{x}(t) = x^i(t)$

$\longrightarrow x^\mu(\lambda) = \begin{pmatrix} ct(\lambda) \\ \vec{x}(\lambda) \end{pmatrix}$.

$x = x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ $\quad \mu = 0,1,2,3 \quad$ $\vec{x} = x^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$i = 1,2,3$

Point in spacetime:

"An event is such a little piece of time-and-space you can mail it through the slotted eye of a cat"
Diane Ackerman, "Mystic Communion of Clocks"

$f(x^\mu(\lambda))$ = scalar function of scalar λ.

$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu} \cdot \frac{dx^\mu}{d\lambda} = (\partial_\mu f)\left(\frac{dx^\mu}{d\lambda}\right) \leftarrow$ "vector"

"covector"

Transform differently.

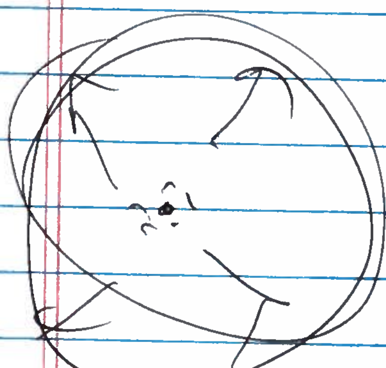
$$\underbrace{\frac{dx^\mu}{d\lambda} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{dx'^\alpha}{d\lambda}}_{\text{chain rule}} \qquad \underbrace{\frac{\partial f}{\partial x^\mu} = \frac{\partial f}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\mu}}$$

$$\frac{df}{d\lambda} = \underline{\frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}} = \left(\frac{\partial f}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\mu}\right)\left(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{dx'^\alpha}{d\lambda}\right)$$

$$= \left(\frac{\partial f}{\partial x'^\beta}\right)\underbrace{\left[\frac{\partial x'^\beta}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\alpha}\right)}_{\frac{\partial x'^\beta}{\partial x'^\alpha} = \delta^\beta_\alpha}\frac{dx'^\alpha}{d\lambda} = \underline{\frac{\partial f}{\partial x'^\alpha}\frac{dx'^\alpha}{d\lambda}}$$

<u>vector</u>: $\frac{dx^\mu}{d\lambda}$, "<u>covector</u>" $\frac{\partial f}{\partial x^\mu} = \partial_\mu f$

transform <u>inversely</u>.



$$\left.\begin{aligned} |\vec{x}|^2 &= c^2t^2 \\ |\vec{x}'|^2 &= c^2t'^2 \end{aligned}\right\} \text{flash expands at } \underline{c} \text{ for two different observers.}$$

$$c^2t^2 - |\vec{x}|^2 = c^2t'^2 - |\vec{x}'|^2 = 0$$

<u>set</u>: $x \cdot y = \eta_{\mu\nu} x^\mu y^\nu$ $\qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\eta = g =$ "<u>metric</u>" ("signature" $+ - - -$)

↑ special

want to find linear transformation

$x' = \Lambda x$. $\boxed{x'^{\alpha} = \Lambda^{\alpha}{}_{\mu} x^{\mu}}$ $\sum_{\mu=0}^{3}$

Such that

$$x' \cdot y' = \eta_{\alpha\beta} x'^{\alpha} y'^{\beta} = \eta_{\alpha\beta} (\Lambda^{\alpha}{}_{\mu} x^{\mu})(\Lambda^{\beta}{}_{\nu} y^{\nu})$$

$$= (\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}) x^{\mu} y^{\nu} = \underbrace{\eta_{\mu\nu} x^{\mu} y^{\nu}}_{= x \cdot y}$$

$$\Rightarrow \boxed{\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}}$$

$$(\Lambda^T)_{\mu}{}^{\alpha} \eta_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu} \qquad \Lambda^T \eta \Lambda = \eta$$

For Λ near identity. $\Lambda^{\alpha}{}_{\mu} = \delta^{\alpha}{}_{\mu} + \varepsilon^{\alpha}{}_{\mu}$

$$\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = \eta_{\alpha\beta} (\delta^{\alpha}{}_{\mu} + \varepsilon^{\alpha}{}_{\mu})(\delta^{\beta}{}_{\nu} + \varepsilon^{\beta}{}_{\nu})$$

$$= \eta_{\alpha\beta} \delta^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu} + \eta_{\alpha\beta} \delta^{\alpha}{}_{\mu} \varepsilon^{\beta}{}_{\nu} + \eta_{\alpha\beta} \varepsilon^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu}$$

$$= \eta_{\mu\nu} + \eta_{\mu\beta} \varepsilon^{\beta}{}_{\nu} + \eta_{\alpha\nu} \varepsilon^{\alpha}{}_{\mu} + O(\varepsilon^2)$$

$$= \cancel{\eta_{\mu\nu}} + \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = \cancel{\eta_{\mu\nu}}$$

$$\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0$$

Antisymmetric ; 4×4 → 6 d.o.f. (x^*)

Basis:

$$(J_1)_{\mu\nu} = \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \quad (J_2)_{\mu\nu} = \begin{pmatrix} & & & \\ & & & -1 \\ & & & \\ & 1 & & \end{pmatrix} \quad (J_3)_{\mu\nu} = \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix}$$

$$(K_1)_{\mu\nu} = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & \\ & & & \end{pmatrix} \quad (K_2)_{\mu\nu} = \begin{pmatrix} & & 1 & \\ & & & \\ -1 & & & \\ & & & \end{pmatrix} \quad (K_3)_{\mu\nu} = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ -1 & & & \end{pmatrix}$$

need $\Lambda^\mu{}_\nu$ $\qquad J^\mu{}_\nu = \eta^{\mu\alpha} J_{\alpha\nu}$ $\qquad \Lambda^\mu{}_\nu = \eta^{\mu\alpha} \Lambda_{\alpha\nu}$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Top row fixed; other three change sign

$$(J_1)^\mu{}_\nu = \begin{pmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \quad (J_2)^\mu{}_\nu = \begin{pmatrix} & & & \\ & & & +1 \\ & & & \\ & -1 & & \end{pmatrix} \quad (J_3) = \begin{pmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{pmatrix}$$

$$(K_1) = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \quad (K_2) = \begin{pmatrix} & & 1 & \\ & & & \\ 1 & & & \\ & & & \end{pmatrix} \quad (K_3) = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}$$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + AJ_1 + BJ_2 + CJ_3 + DK_1 + EK_2 + FK_3$$

$$\boxed{\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + (\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})^\mu{}_\nu}$$ infinitesimal

finite $\rightarrow$ $$\Lambda^\mu{}_\nu = \lim_{N\to\infty}\left(\delta^\mu_\nu + \frac{1}{N}(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})\right)^N = \underline{\exp(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})}$$

$$J_1 = \left(\begin{array}{cc|cc} & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right) \qquad J_1^2 = \left(\begin{array}{cc|cc} & & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right)$$

$$J_1^3 = \left(\begin{array}{cc|cc} & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right) = -J_1 \qquad J_1^4 = \left(\begin{array}{cc|cc} & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

$$\exp(\theta J_1) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \theta\left(\begin{array}{cc|cc} & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right) + \frac{1}{2}\theta^2\left(\begin{array}{cc|cc} & & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right)$$

$$+ \frac{1}{6}\theta^3\left(\begin{array}{cc|cc} & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right) + \frac{1}{24}\theta^4\left(\begin{array}{cc|cc} & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) + \cdots$$

$$= \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots & -\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \cdots\right) \\ & & \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \cdots & 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{array}\right)$$

x-axis **rotation**

$$K_1 = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & \\ & & & \end{array}\right) \qquad K^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{pmatrix} \qquad K^3 = K \quad \underline{\text{etc.}}$$

$$\exp(\zeta K_1) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \zeta\left(\begin{array}{c|c} K_1 & \\ \hline & \end{array}\right) + \frac{1}{2}\zeta^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{pmatrix} + \cdots$$

$$= \left(\begin{array}{cc|cc} 1 + \frac{1}{2}\zeta^2 + \frac{1}{24}\zeta^4 + \cdots & \zeta + \frac{1}{6}\zeta^3 + \frac{1}{120}\zeta^5 + \cdots & & \\ \zeta + \frac{1}{6}\zeta^3 + \frac{1}{120}\zeta^5 + \cdots & 1 + \frac{1}{2}\zeta^2 + \frac{1}{24}\zeta^4 + \cdots & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

$$\exp(\zeta K_1) = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\underbrace{\begin{aligned} ct' &= \cosh\zeta \cdot ct + \sinh\zeta\, x \\ x' &= \sinh\zeta \cdot ct + \cosh\zeta\, x \end{aligned}}$$

$x = 0$

$$\left.\begin{aligned} x' &= \sinh\zeta \cdot ct \\ ct' &= \cosh\zeta \cdot ct \end{aligned}\right\} \quad \frac{x' = (c\tanh\zeta)\, t'}{x' = v\, t'}$$

at rest in O. $\rightarrow$ moving at speed v in O'

$$\frac{v}{c} = \tanh \zeta$$

$$\cosh^2\zeta - \sinh^2\zeta = 1$$

$$1 - \tanh^2\zeta = \frac{1}{\cosh^2\zeta} = 1 - v^2/c^2$$

$$\cosh\zeta = \frac{1}{\sqrt{1-v^2/c^2}} \qquad \sinh\zeta = \frac{v/c}{\sqrt{1-v^2/c^2}}$$

$$\left.\begin{aligned} ct' &= \gamma\left(ct + \frac{v}{c}x\right) \\ x' &= \gamma\left(x + \frac{v}{c}\cdot ct\right) \end{aligned}\right\} \quad \boxed{\begin{aligned} t' &= \gamma\left(t + \frac{vx}{c^2}\right) \\ x' &= \gamma(x + vt) \end{aligned}}$$