4/2/2018

$$\underline{x} = x^{\mu} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ x^{i} \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} ct \\ \underset{\sim}{x} \end{pmatrix}$$

$\underline{\mu = 0,1,2,3}$ $\qquad$ $\underline{i = 1,2,3}$.

$$x \cdot y = (x^0)(y^0) - (x^1)(y^1) - (x^2)(y^2) = (x^3)(y^3)$$

$$= x^{\mu} y^{\nu} \eta_{\mu\nu} = (x)^T \eta y = (y)^T \eta x$$

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$x \cdot y = (x^{\mu} \eta_{\mu\nu} y^{\nu}) = x_{\nu} y^{\nu} = (x^{\mu})(\eta_{\mu\nu} y^{\nu}) = x^{\mu} y_{\mu}$$

η "lowers index".

$\underline{x_{\mu} = \eta_{\mu\nu} x^{\nu}}$ $\qquad$ $x^{\mu} = (\eta^{-1})^{\mu\nu} x_{\nu} = \eta^{\mu\nu} x_{\nu}$

$$\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$X^2 = x \cdot x = (ct)^2 - |\vec{x}|^2$ $\qquad$ $x \cdot x = 0$ $\rightarrow$ speed of light same..

$= x' \cdot x'$

2

Want to find linear transformation

$x' = \Lambda x$. $\boxed{x'^{\alpha} = \Lambda^{\alpha}{}_{\mu} x^{\mu}}$ $\sum_{\mu=0}^{3}$

Such that

$x' \cdot y' = \eta_{\alpha\beta} x'^{\alpha} y'^{\beta} = \eta_{\alpha\beta} (\Lambda^{\alpha}{}_{\mu} x^{\mu})(\Lambda^{\beta}{}_{\nu} y^{\nu})$

$= (\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}) x^{\mu} y^{\nu} = \underbrace{\eta_{\mu\nu} x^{\mu} y^{\nu}}_{= x \cdot y}$

$\Rightarrow \boxed{\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}}$

$(\Lambda^T)_{\mu}{}^{\alpha} \eta_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}$ $\quad \Lambda^T \eta \Lambda = \eta$

For Λ near identity, $\Lambda^{\alpha}{}_{\mu} = \delta^{\alpha}{}_{\mu} + \varepsilon^{\alpha}{}_{\mu}$

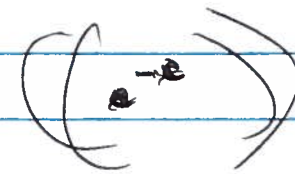
$\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = \eta_{\alpha\beta} (\delta^{\alpha}{}_{\mu} + \varepsilon^{\alpha}{}_{\mu})(\delta^{\beta}{}_{\nu} + \varepsilon^{\beta}{}_{\nu})$

$= \eta_{\alpha\beta} \delta^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu} + \eta_{\alpha\beta} \delta^{\alpha}{}_{\mu} \varepsilon^{\beta}{}_{\nu} + \eta_{\alpha\beta} \varepsilon^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu}$

$= \eta_{\mu\nu} + \eta_{\mu\beta} \varepsilon^{\beta}{}_{\nu} + \eta_{\alpha\nu} \varepsilon^{\alpha}{}_{\mu} + O(\varepsilon^2)$

$= \cancel{\eta_{\mu\nu}} + \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = \cancel{\eta_{\mu\nu}}$

$$\boxed{\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0}$$

Antisymmetric ; 4x4 $\rightarrow$ 6 d.o.f. 

Basis :

$$(J_1)_{\mu\nu} = \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \quad (J_2)_{\mu\nu} = \begin{pmatrix} & & & \\ & & & -1 \\ & & & \\ & 1 & & \end{pmatrix} \quad (J_3)_{\mu\nu} = \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix}$$

$$(K_1)_{\mu\nu} = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & \\ & & & \end{pmatrix} \quad (K_2)_{\mu\nu} = \begin{pmatrix} & & 1 & \\ & & & \\ -1 & & & \\ & & & \end{pmatrix} \quad (K_3)_{\mu\nu} = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ -1 & & & \end{pmatrix}$$

need $\Lambda^{\mu}{}_{\nu}$ $\quad J^{\mu}{}_{\nu} = \eta^{\mu\alpha} J_{\alpha\nu}$ $\quad \Lambda^{\mu}{}_{\nu} = \eta^{\mu\alpha} \Lambda_{\alpha\nu}$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Top row fixed : other three change sign

$$(J_1)^{\mu}{}_{\nu} = \begin{pmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \quad (J_2)^{\mu}{}_{\nu} = \begin{pmatrix} & & & \\ & & & +1 \\ & & & \\ & -1 & & \end{pmatrix} \quad (J_3) = \begin{pmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{pmatrix}$$

$$(K_1) = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \quad (K_2) = \begin{pmatrix} & & 1 & \\ & & & \\ 1 & & & \\ & & & \end{pmatrix} \quad (K_3) = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}$$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + AJ_1 + BJ_2 + CJ_3 + DK_1 + EK_2 + FK_3$$

$$\boxed{\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)^\mu{}_\nu}$$ infinitesimal

finite $\rightarrow \Lambda^\mu{}_\nu = \lim_{N\to\infty}\left(\delta^\mu{}_\nu + \frac{1}{N}\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)\right)^N = \exp\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)$

$$J_1 = \begin{pmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \qquad J_1^2 = \begin{pmatrix} & & & \\ & & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$J_1^3 = \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} = -J_1 \qquad J_1^4 = \begin{pmatrix} & & & \\ & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\exp(\theta J_1) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \theta \begin{pmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} + \frac{1}{2}\theta^2 \begin{pmatrix} & & & \\ & & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$+ \frac{1}{6}\theta^3 \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} + \frac{1}{24}\theta^4 \begin{pmatrix} & & & \\ & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \cdots$$

$$= \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4+\cdots & -\left(\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5+\cdots\right) \\ & & \theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5+\cdots & 1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4+\cdots \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{array}\right)$$

x axis rotation

$$K_1 = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & \\ & & & \end{array}\right) \qquad K^2 = \begin{pmatrix} 1 & \\ & 1 \\ & & \\ & & \end{pmatrix} \qquad K^3 = K \quad \underline{\text{etc.}}$$

$$\exp(\zeta K_1) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \zeta \left(\begin{array}{cc|cc} & \zeta & & \\ \zeta & & & \\ \hline & & & \\ & & & \end{array}\right) + \frac{1}{2}\zeta^2 \begin{pmatrix} 1 & \\ & 1 \\ & & \\ & & \end{pmatrix} + \cdots$$

$$= \left(\begin{array}{cc|cc} 1 + \frac{1}{2}\zeta^2 + \frac{1}{24}\zeta^4 + \cdots & \zeta + \frac{1}{6}\zeta^3 + \frac{1}{120}\zeta^5 + \cdots & & \\ \zeta + \frac{1}{6}\zeta^3 + \frac{1}{120}\zeta^5 + \cdots & 1 + \frac{1}{2}\zeta^2 + \frac{1}{24}\zeta^4 + \cdots & & \\ & & 1 & \\ & & & 1 \end{array}\right)$$

$$\exp(\zeta K_1) = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \begin{aligned} ct' &= \cosh\zeta \cdot ct + \sinh\zeta\, x \\ x' &= \sinh\zeta \cdot ct + \cosh\zeta\, x \end{aligned}$$

$x = 0$

$$\left.\begin{aligned} x' &= \sinh\zeta \cdot ct \\ ct' &= \cosh\zeta \cdot ct \end{aligned}\right\} \quad \frac{x' = (c \tanh\zeta)\, t'}{x' = v\, t'}$$

at rest in O. $\rightarrow$ moving at speed v in O'

$$\frac{v}{c} = \tanh \zeta \qquad \cosh^2\zeta - \sinh^2\zeta = 1$$

$$1 - \tanh^2\zeta = \frac{1}{\cosh^2\zeta} = 1 - v^2/c^2$$

$$\cosh\zeta = \frac{1}{\sqrt{1-v^2/c^2}} \qquad \sinh\zeta = \frac{v/c}{\sqrt{1-v^2/c^2}}$$

$$ct' = \gamma\left(ct + \frac{v}{c}x\right)$$
$$x' = \gamma\left(x + \frac{v}{c}\cdot ct\right)$$

$$t' = \gamma\left(t + \frac{vx}{c^2}\right)$$
$$x' = \gamma(x + vt)$$

Properties · $\eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = (\Lambda^T)\eta\Lambda = \eta_{\mu\nu} = \eta$

$$\rightarrow \quad (\det\Lambda)^2 = 1 \qquad \det\Lambda = \pm 1$$

$$\eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu} \qquad \mu = \nu = 0$$

$$\eta_{\alpha\beta}\Lambda^\alpha{}_0 \Lambda^\beta{}_0 = \eta_{00}(\Lambda^0{}_0)^2 + \sum_i \eta_{ii}(\Lambda^i{}_0)^2 = \eta_{00} = 1$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2$$

$$\Lambda^0{}_0 \geq 1$$
$$\Lambda^0{}_0 \leq -1$$

$\Lambda = \exp\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)$

$\det M = \lambda_1 \cdots \lambda_N \qquad \log \det M = \log \lambda_1 + \cdots + \log \lambda_N \quad = \mathrm{Tr}(\ln M)$

$$\boxed{\det M = \exp\left(\mathrm{Tr}(\ln M)\right)}$$

$\det \Lambda = \exp\left[\mathrm{Tr}\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)\right] = e^0 = 1.$

$\exp\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)$ continuously connected to $\mathbb{1}$

$\rightarrow \Lambda^0{}_0$ can't change sign $\rightarrow \boxed{\Lambda^0{}_0 \geq 1}$

$\boxed{(\cosh \zeta \geq 1)}$

$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad PT = TP = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$

$\det = -1 \qquad \det = -1 \qquad \det(PT) = 1$

$T^0{}_0 = -1 \qquad P^0{}_0 = +1 \qquad (PT)^0{}_0 = -1.$

Four disconnected components:

Proper Λ

(Proper) $\times T$ $\qquad$ ( ) $\times P$ $\qquad$ ( ) $\times$ (PT)

$\Lambda^T \eta \Lambda = 1$. $\quad \eta^{-1}\Lambda^T \eta \Lambda = 1$ $\quad \boxed{\eta^{-1}\Lambda^T \eta = \Lambda^{-1}}$

$$(\Lambda^{-1})^{\alpha}{}_{\beta} = \eta^{\alpha\mu} (\Lambda^T)_{\mu}{}^{\nu} \cdot \eta_{\nu\beta}$$

$$= \eta^{\alpha\mu} \Lambda^{\nu}{}_{\mu}\, \eta_{\nu\beta}$$

$$= \eta_{\beta\nu} \Lambda^{\nu}{}_{\mu} \cdot \eta^{\mu\alpha} = \Lambda_{\beta}{}^{\alpha}$$

Rotation lives in (ij) $\quad \Lambda = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ $\quad \Lambda^2 = 1$.

$\boxed{R^{-1} = R^T}$ ✓ $\quad \boxed{\theta \to -\theta}$

Boost. $\quad \Lambda^T = \Lambda, \quad \eta\Lambda\eta \to \left(\begin{array}{c|c} + & - \\ \hline - & \cancel{(\ )^2} \end{array}\right)$ $\Rightarrow$ $\boxed{v \to -v}$ $\quad \varphi \to -\varphi$