4|4|2018 **Lorentz Transformation.**

$$x'^{\mu} = \Lambda^{\mu}{}_{\alpha} x^{\alpha}.$$

$$\boxed{\begin{aligned} (ct)' &= \gamma\left(ct + \frac{v}{c}\cdot x\right). \\ x' &= \gamma\left(x + \frac{v}{c}\cdot ct\right) \end{aligned}}$$

Properties:

$$\eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}.$$

$$\Longleftrightarrow \quad (\Lambda^{T})\eta\Lambda = \eta.$$

$$\Longrightarrow \quad (\det\Lambda)^{2} = 1. \qquad \det\Lambda = \pm 1$$

$$\eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu} \qquad \text{Take } \mu = \nu = 0$$

$$\eta_{\alpha\beta}\Lambda^{\alpha}{}_{0}\Lambda^{\beta}{}_{0} = \underbrace{\eta_{00}}_{1}(\Lambda^{0}{}_{0})^{2} + \sum_{i}\underbrace{\eta_{ii}}_{-1}(\Lambda^{i}{}_{0})^{2}$$

$$\boxed{(\Lambda^{0}{}_{0})^{2} = 1 + \sum_{i}(\Lambda^{i}{}_{0})^{2}} \qquad \begin{aligned} \Lambda^{0}{}_{0} &\geq 1 \\ \Lambda^{0}{}_{0} &\leq -1 \end{aligned}$$

$\Lambda = \exp(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})$

$\det M = \lambda_1 \cdots \lambda_N$ $\qquad \log\det M = \log\lambda_1 + \cdots + \log\lambda_N = \mathrm{Tr}(\ln M)$

$$\boxed{\det M = \exp(\mathrm{Tr}(\ln M))}$$

$\det\Lambda = \exp[\mathrm{Tr}(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})] = e^0 = 1.$

$\exp(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})$ continuously connected to $\mathbb{1}$

$\to \Lambda^0{}_0$ can't change sign $\to$ $\Lambda^0{}_0 \geq 1$

$(\cosh\zeta \geq 1)$ $\qquad$ (works for det. also)

$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\quad P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\quad PT = TP = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$

$\det = -1$ $\qquad \det = -1$ $\qquad \det(PT) = 1$

$T^0{}_0 = -1$ $\qquad P^0{}_0 = +1$ $\qquad (PT)^0{}_0 = -1.$

Four disconnected components:

Proper Λ

(Proper) $\times T$ $\qquad$ ( ) $\times P$ $\qquad$ ( ) $\times (PT)$

$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$ = rotation 180° z-axis. $\qquad \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $T \times$ Rotation

$$\underline{\Lambda^T \eta \Lambda = 1} \cdot \quad \underline{\eta^{-1}\Lambda^T \eta \Lambda = 1} \quad \boxed{\underline{\eta^{-1}\Lambda^T \eta = \Lambda^{-1}}}$$

$$(\Lambda^{-1})^{\alpha}{}_{\beta} = \eta^{\alpha\mu} (\Lambda^T)_{\mu}{}^{\nu} \cdot \eta_{\nu\beta}$$

$$= \eta^{\alpha\mu} \Lambda^{\nu}{}_{\mu} \eta_{\nu\beta}$$

$$= \underline{\eta_{\beta\nu} \Lambda^{\nu}{}_{\mu} \cdot \eta^{\mu\alpha} = \Lambda_{\beta}{}^{\alpha}}$$

<u>Rotation</u> lives in $\underline{(ij)}$ $\quad \Lambda = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \Lambda^2 = 1.$

$$\boxed{R^{-1} = R^T} \quad \checkmark \qquad \boxed{\theta \to -\theta}$$

Boost. $\quad \Lambda^T = \Lambda, \quad \eta \Lambda \eta \to \left(\begin{array}{c|c} + & - \\ \hline - & (\cancel{\ })^2 \end{array}\right) \quad \Rightarrow \boxed{v \to -v} \quad \boxed{\varphi \to -\varphi}$

$$\cancel{x'^{\mu} = x^{\mu}{}_{\nu} x^{\nu}}$$

$$\underline{x_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu}} \qquad y^{\mu} = \Lambda^{\mu}{}_{\rho} y^{\rho}$$

$$\cancel{x_{\mu} y^{\mu} = (\Lambda_{\mu}{}^{\nu} x)(\Lambda}$$

$$x_{\mu} y^{\mu} = (\Lambda_{\mu}{}^{\nu} x_{\nu})(\Lambda^{\mu}{}_{\rho} y^{\rho})$$

$$= x_{\nu} (\Lambda_{\mu}{}^{\nu} \Lambda^{\mu}{}_{\rho}) y^{\rho} = \cancel{x_{\rho} y^{\rho}}$$

$$\underline{\Lambda^{-1\,\nu}{}_{\mu} \Lambda^{\mu}{}_{\rho} = \delta^{\nu}{}_{\rho}.}$$

Tensors. $x'^{\alpha} = \Lambda^{\alpha}{}_{\mu} x^{\mu}$ vector

$$T'^{\alpha\beta}{}_{\gamma\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \Lambda_{\gamma}{}^{\rho} \Lambda_{\delta}{}^{\sigma} T^{\mu\nu}{}_{\rho\sigma}.$$

(m,n) tensor: m superscript, n subscript.
order matters.

Since η can raise, lower any index. $T^{\alpha\beta\gamma\delta}$ rank $= r$

Special Tensors: $\boxed{\eta'_{\alpha\beta} = \Lambda_{\alpha}{}^{\mu} \Lambda_{\beta}{}^{\nu} \eta_{\mu\nu} = \eta_{\alpha\beta}}$

$\varepsilon^{0123} = +1$ (MM • 139) $\varepsilon_{0123} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\varepsilon^{0123} = -1$

$$\varepsilon'^{\alpha\beta\gamma\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \Lambda^{\gamma}{}_{\rho} \Lambda^{\delta}{}_{\sigma} \cdot \varepsilon^{\mu\nu\rho\sigma}$$

$$= (\det \Lambda)\,(\varepsilon^{\alpha\beta\gamma\delta}) = \varepsilon^{\alpha\beta\gamma\delta}.$$ (proper)

Traces $T^{\mu}{}_{\nu} = T^{\mu\alpha}{}_{\alpha\nu}$.

$$T' = \Lambda\Lambda M T$$

$$\mathrm{Tr}\,T' = \mathrm{Tr}(\Lambda \underline{M} \Lambda T) = \Lambda\Lambda\,\mathrm{Tr}\,T.$$

before or after

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$$T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) \qquad T_{(\alpha\beta\gamma)} \quad \frac{(n+s-1)!}{n!\,(s-1)!} \quad \circ|\circ\circ|\circ\circ$$

$$T_{[\alpha\beta]} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) \qquad T_{[\alpha\beta\gamma]} : \binom{n}{a} = \frac{n!}{a!\,(n-a)!}$$

$$T_{(\alpha\beta)} + T_{[\alpha\beta]} = T_{\alpha\beta}$$

$$T_{(\alpha\beta\gamma)} + T_{[\alpha\beta\gamma]} \neq T_{\alpha\beta\gamma}$$

$$\frac{4\cdot5\cdot\cancel{6}}{\cancel{6}} = 20 \qquad \frac{4\cdot3\cdot2}{6} = 4 \qquad 4^3 = 64$$

Dual,

$$(*V)_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\mu} V^{\mu}$$

$$(*V)_{\alpha} = \frac{1}{3!}\varepsilon_{\alpha\mu\nu\rho} V^{[\mu\nu\rho]}$$

$$(*T)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu} T^{[\mu\nu]}$$

E.M. field:

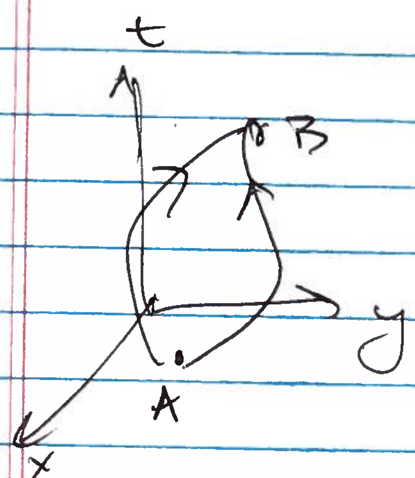
$x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ is a vector

$\vec{v} = \frac{d\vec{x}}{dt}$ $\quad v^\mu = \frac{dx^\mu}{dt}$ not a vector.

$$\frac{d\vec{x}'}{dt'} = \frac{\cancel{\gamma}(d\vec{x} + \vec{v}\,dt)}{\cancel{\gamma}(dt + \frac{\vec{v}\cdot d\vec{x}}{c^2})} = \frac{\left(\frac{d\vec{x}}{dt} + \vec{v}\right)}{\left(1 + \frac{\vec{v}\cdot \frac{d\vec{x}}{dt}}{c^2}\right)},$$

$$dt^2 - \frac{1}{c^2}(d\vec{x})^2 = \frac{1}{c^2}dx\cdot dx = \left(1 - \frac{v^2}{c^2}\right)dt^2 = d\tau^2$$

is a scalar. $\boxed{d\tau = \frac{1}{\gamma}dt}$



$\tau_{AB} = \int_A^B d\tau$

is path-dependent,

("twin paradox")