4/25/2018 $S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$. ϕ_i A_ν

$\delta S = 0$ $\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$ (12.83)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0.$$

$$\mathcal{L} = \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - V \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$$

$$\partial^\mu \partial_\mu \phi = \Box^2 \phi = -\frac{\partial V}{\partial \phi}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}.$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V\right)$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V = u = \varepsilon = \rho.$$

(kinetic) + (strain) + (potential)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

stress-energy tensor $= \frac{\omega^2 |\phi|^2}{2}$

Plane wave $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ $\rho = \frac{1}{2}\omega^2 |\phi|^2 + \frac{1}{2}|\vec{k}|^2 |\phi|^2$

$$\vec{S} = \omega \vec{k} |\phi|^2$$

(2)

$$T^{\mu\nu} = \partial^\mu\phi \cdot \partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\partial_\mu T^{\mu\nu} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi\right) - \eta^{\mu\nu}\partial_\mu\mathcal{L}$$

$$= \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\cdot\partial^\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot(\partial_\mu\partial^\nu\phi)\right.$$

$$\left. - \eta^{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\partial_\mu(\partial_\alpha\phi)\right)\right)$$

$$= \cancel{\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)}\cdot\partial^\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial^\nu(\partial_\mu\phi)$$

So.

$$- \cancel{\frac{\partial\mathcal{L}}{\partial\phi}}\cdot\partial^\nu\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\cdot\partial^\nu(\partial_\alpha\phi).$$

$T^{00} = \rho$ $\quad$ $T^{i0} = \vec{S}$ $\quad$ $\partial_0 T^{00} + \partial_i T^{i0}$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V. \qquad \vec{S} = \dot{\phi}\,\partial^i\phi = -\dot{\phi}\,\vec{\nabla}\phi$$

$$\frac{\partial\rho}{\partial t} = \dot{\phi}\ddot{\phi} + \vec{\nabla}\phi\cdot\vec{\nabla}\dot{\phi} + \frac{\partial V}{\partial\phi}\dot{\phi} - (\vec{\nabla}\dot{\phi}\cdot\vec{\nabla}\phi + \dot{\phi}\,\nabla^2\phi)$$

$$+\vec{\nabla}\cdot\vec{S}$$

$$= \dot{\phi}\left(\cancel{\ddot{\phi} - \nabla^2\phi + \frac{\partial V}{\partial\phi}}\right) + \vec{\nabla}\phi\cdot\vec{\nabla}\dot{\phi} - \vec{\nabla}\dot{\phi}\cdot\vec{\nabla}\phi \;= 0. \checkmark$$

Elm: Vector field A^μ

① "KE" $\dot{A}^2 \to \partial_\mu A_\nu \to F_{\mu\nu}$ [gauge invariant]

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}} \quad (12.85)$$

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F_{\rho\sigma} \eta^{\rho\alpha} \eta^{\sigma\beta}$$

$$= -\frac{1}{16\pi} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \eta^{\rho\alpha} \eta^{\sigma\beta}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{16\pi} \Big(\alpha=\mu, \beta=\nu \; + \; \beta=\mu, \alpha=\nu$$

$$+ \; \rho=\mu, \sigma=\nu \; + \; \rho=\nu, \sigma=\mu \Big)$$

$$= -\frac{1}{16\pi} \Big(F_{\rho\sigma} \eta^{\rho\mu} \eta^{\sigma\nu} - F_{\rho\sigma} \eta^{\rho\nu} \eta^{\sigma\mu}$$

$$+ F_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta} - F_{\alpha\beta} \eta^{\nu\alpha} \eta^{\mu\beta} \Big)$$

$$= -\frac{1}{16\pi} \left(F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \right) = -\frac{1}{4\pi} F^{\mu\nu}$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu\left(-\frac{1}{4\pi}F^{\mu\nu}\right) = \frac{\partial \mathcal{L}}{\partial(A_\nu)} = -\frac{1}{c}J^\nu .$$

$$\boxed{\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\nu J^\nu} \quad (12.85)$$

Term $A_\nu J^\nu$ → source for A_ν ✓

also : interaction of matter with E/M.

(3)

<u>Symmetries</u>. $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi}) \cdot (\partial^\mu \vec{\phi}) - V(\vec{\phi} \cdot \vec{\phi})$$

$$= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1^2 + \phi_2^2)$$

$\vec{\phi}' = R\vec{\phi}$ $\qquad \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

~~Let~~ $\dfrac{d\mathcal{L}}{d\alpha} = 0$ $\qquad$ <u>symmetry</u>.

<u>Let</u> $\quad J^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\phi})} \cdot \dfrac{d\vec{\phi}}{d\alpha} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \dfrac{d\phi_j}{d\alpha} \qquad \sum_{j=1}$

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \frac{d\phi_j}{d\alpha} \right)$$

$$= \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \right) \left(\frac{d\phi_j}{d\alpha} \right) + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \right) \partial_\mu \left(\frac{d\phi_j}{d\alpha} \right)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_j} \cdot \frac{d\phi_j}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \frac{d}{d\alpha}(\partial_\mu \phi_j)$$

$$= \frac{d\mathcal{L}}{d\alpha} = 0 .$$

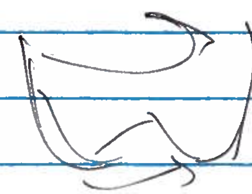
Symmetry $\longleftrightarrow$ conserved current

conserved charge $= \int d^3x \, J^0$

Nöther's Theorem — most important result of 20th century Physics

Broken symmetry:

$$\boxed{V = \frac{1}{4}\lambda\left(\vec{\phi}\cdot\vec{\phi} - a^2\right)^2}$$



minimum energy configuration

$\vec{\phi}$: constant. $|\vec{\phi}| = a$ $\quad \partial\vec{\phi} = 0 \quad u = 0$.

$\vec{\phi} \in M_0 = \{ V(\vec{\phi}) = 0 \}$. Let $\phi_2 = a + \delta\phi_2$. $\phi_1 = 0 + \delta\phi_1$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \delta\vec{\phi})\cdot(\partial^\mu \delta\vec{\phi}) - \frac{1}{4}\lambda\Big(\delta\phi_1^2 + \underbrace{(a+\delta\phi_2)^2 - a^2}_{\cancel{a^2} + 2a\delta\phi_2}\Big)^2$$

$$= \frac{1}{4}\lambda(2a\,\delta\phi_2)^2 + O(\delta\phi^3)$$

$m_1 = 0 \qquad \frac{1}{2}m_2^2(\delta\phi_2)^2 = \frac{1}{4}\lambda\cdot 4a^2 \cancel{\delta\phi_2^2} \qquad m_2^2 = 2\lambda a^2$

residual symmetry $\longleftrightarrow$ "massless excitation"

"Goldstone Boson"

Complex ϕ.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi^* \phi)$$

$$\left.\begin{aligned} \phi &= \phi_1 + i\phi_2 \\ \phi^* &= \phi_1 - i\phi_2 \end{aligned}\right\} \quad \begin{aligned} \phi &\to e^{i\alpha}\phi \\ \phi^* &\to e^{-i\alpha}\phi^* \end{aligned} \quad \longleftrightarrow \text{Rotation.}$$

$$\mathcal{L} = \frac{1}{2}|\partial\phi|^2 - V|\phi|^2 \qquad \text{independent of } \alpha.$$

$U(1) \longleftrightarrow SO(2)$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{d\phi}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\frac{d\phi^*}{d\alpha}$$

$$= \frac{1}{2}(\partial^\mu \phi)(i\phi) + \frac{1}{2}(\partial^\mu \phi)(-i\phi^*)$$

$$\boxed{J^\mu = \frac{i}{2}\left(\partial^\mu \phi^* \cdot \phi - \phi^* \cdot \partial^\mu \phi\right)}$$

suppose: $\alpha \to \alpha(x)$. $\quad |\phi|^2$ ✓

$$\partial_\mu(e^{i\alpha}\phi) = e^{i\alpha}(\partial_\mu \phi) + i(\partial_\mu \alpha)\phi$$

gradient of α appears.

Recall: gauge transformation shifts A_μ by gradient

Let $\boxed{D_\mu = \partial_\mu - ieA_\mu}$ "covariant derivative"

Gauge transformation: $\phi \to \phi' = e^{ie\alpha}\phi$

$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$.

leaves $|\phi|^2$ $F_{\mu\nu}$ unchanged.

$$D'_\mu \phi' = (\partial_\mu - ie(A_\mu + \partial_\mu \alpha))(e^{ie\alpha}\phi)$$

$$= e^{ie\alpha}\partial_\mu \phi + \cancel{ie\,\partial_\mu\alpha\, e^{ie\alpha}\phi}$$

$$- ieA_\mu e^{ie\alpha}\phi - \cancel{ie\,\partial_\mu\alpha\, e^{ie\alpha}\phi}$$

$$= e^{ie\alpha}(D_\mu \phi).$$

$$\boxed{\mathcal{L} = \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^*\phi)}$$

invariant.

J^μ = Noether current $= \dfrac{\partial \mathcal{L}}{\partial A_\mu}$