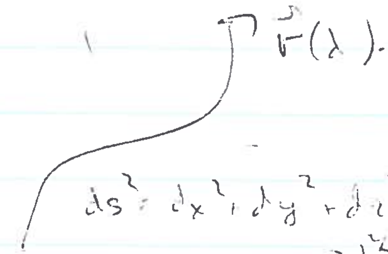
1/22/2016. Fermat's principle! (Action) = (time)

$v = c/n \quad t = d/v \quad nd/c$

$$S = \int n\, ds$$

$\vec r(\lambda)$

$$= \int d\lambda \; n(\vec r(\lambda)) \sqrt{\left(\frac{d\vec r}{d\lambda}\right)^2}$$



$ds^2 = dx^2 + dy^2 + dz^2$

$= \left[\left(d\vec r/d\lambda\right)^2\right] d\lambda^2$

reparametrization invariant. $\lambda = \lambda(\mu)$

$$S = \int \left(\frac{d\lambda}{d\mu}\right) d\mu \; n(\vec r(\lambda(\mu))) \sqrt{\left(\frac{d\vec r}{d\mu}\frac{d\mu}{d\lambda}\right)^2}$$

$$= \int d\mu \; n(\vec r(\mu)) \sqrt{\left(\frac{d\vec r}{d\mu}\right)^2}$$

$$\delta S = \int d\lambda \; n(\vec r + \delta\vec r)\left(\left[\frac{d}{d\lambda}(\vec r + \delta \vec r)\right]^2\right)^{1/2} - \int d\lambda \; n \sqrt{\left(\frac{d\vec r}{d\lambda}\right)^2}$$

$$= \int d\lambda \left\{ (\vec\nabla n \cdot \delta\vec r) \sqrt{\left(\frac{d\vec r}{d\lambda}\right)^2} - \frac{n}{\sqrt{(d\vec r/d\lambda)^2}} \frac{d\vec r}{d\lambda} \cdot \frac{d}{d\lambda}(\delta\vec r) \right\}$$

$$= \int d\lambda \left\{ \vec\nabla n \cdot \sqrt{\left(\frac{d\vec r}{d\lambda}\right)^2} - \frac{d}{d\lambda}\left[\frac{n(\vec r)}{\sqrt{(d\vec r/d\lambda)^2}} \frac{d\vec r}{d\lambda}\right] \right\} \cdot \delta\vec r$$

$$\vec{\nabla} n \sqrt{\left(\frac{d\vec{r}}{d\lambda}\right)^2} - \frac{d}{d\lambda}\left(\frac{n}{\sqrt{(d\vec{r}/d\lambda)^2}}\frac{d\vec{r}}{d\lambda}\right) = 0,$$

$$\left(\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = 0\right)$$

$$\left(\frac{d\vec{r}}{d\lambda}\right)^2 = \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2 = \left(\frac{ds}{d\lambda}\right)^2$$

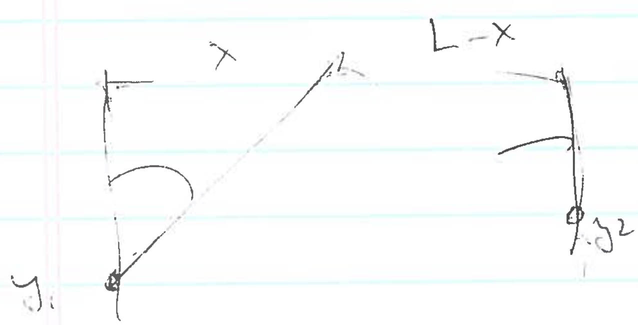
choose $\lambda = s$ invariant!

$$\boxed{\frac{d}{ds}\left[n(\vec{r})\frac{d\vec{r}}{ds}\right] = \vec{\nabla} n} \quad (8.114).$$

n = constant $\vec{\nabla} n = 0$.

$$\frac{d^2\vec{r}}{ds^2} = 0 \qquad \frac{d\vec{r}}{ds} = \vec{t} \qquad \vec{r} = \vec{r}_0 + \vec{t}s$$

straight line.



$$S = n\sqrt{x^2+y_1^2} + n\sqrt{(L-x)^2+y_2^2}$$

$$\frac{\partial S}{\partial x} = n\frac{x}{\sqrt{x^2+y_1^2}} - n\frac{(L-x)}{\sqrt{(L-x)^2+y_2^2}} = 0$$

$$\frac{x}{\sqrt{x^2+y_1^2}} = \frac{(L-x)}{\sqrt{(L-x)^2+y_2^2}}$$

$\sin\theta_1 = \sin\theta_2 \qquad \theta_1 = \theta_2$

$(n_1 \sin\theta_1 = n_2 \sin\theta_2)$

(3)

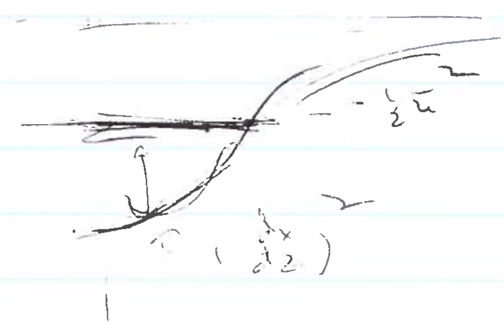
$\frac{\partial u}{\partial z} = 0$ $\quad \frac{d}{dz}\left(u(x) \cdot \frac{dz}{ds}\right) = 0 \qquad u(x)\cos\theta = u_0 \cos\theta_0 = \bar{u}$

$$\bar{u}^2 \frac{d^2x}{dz^2} = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) \qquad F = ma \quad (8.116)$$

$$\frac{1}{2}\bar{u}^2\left(\frac{dx}{dz}\right)^2 - \frac{1}{2}u^2 = -\frac{1}{2}\bar{u}^2$$

Energy conservation (8.117)

$$V = -\frac{1}{2}u^2(x)$$



$$\left(\frac{dx}{dz}\right)^2 = \frac{u^2 - \bar{u}^2}{\bar{u}^2} \rightarrow \frac{dx}{dz} = \frac{\bar{u}}{\sqrt{u^2 - \bar{u}^2}}$$

(8.118)

($\frac{1}{4}$ period) $\quad \int dz = Z_1 = \int \frac{dz}{dx}\, dx = \int_0^{x_{max}} \frac{\bar{u}\, dx}{\sqrt{u^2 - \bar{u}^2}}$

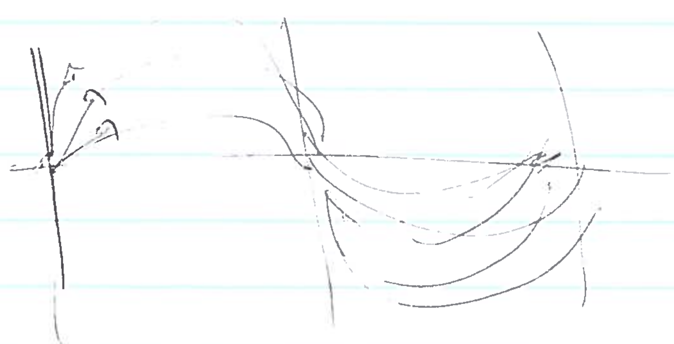
path length: $ds^2 = dx^2 + dz^2 = dx^2\left(1 + \left(\frac{dz}{dx}\right)^2\right)$.

$$L = \int ds = \int dx \sqrt{1 + \left(\frac{dz}{dx}\right)^2} = \int dx \left(1 + \frac{\bar{n}^2}{n^2 - \bar{n}^2}\right)^{\frac{1}{2}}$$

$$= \int_0^{x_{max}} \frac{n(x)\,dx}{\sqrt{n^2 - \bar{n}^2}} = L_{physical}$$

phase $e^{i\frac{\omega}{c}S}$ $\quad dS = \vec{\nabla} S \cdot d\vec{r} = n\hat{k} \cdot d\vec{r} = n\,ds$

$$\boxed{L_{optical} = \int_0^{x_{max}} \frac{n^2(x)\,dx}{\sqrt{n^2 - \bar{n}^2}}} \quad (8.119)$$



If you were very lucky, Z independent of θ_0
$L_{opt.}$ independent of θ_0
arrive together, in phase.

(5)

Cylindrical (fibers). x, y, z (θ, ϕ)

$$\hat{k} = \frac{d\vec{r}}{ds} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\boxed{n = n(\rho)} \quad \frac{\partial n}{\partial z} = 0 \rightarrow \frac{d}{ds}\left(n\frac{dz}{ds}\right) = \frac{\partial n}{\partial z} = 0$$

$$n\cos\theta = n_0\cos\theta_0 = \bar{n}$$

$$\bar{n}^2\frac{d^2x}{dz^2} = \frac{\partial}{\partial x}\left(\frac{1}{2}n^2\right) = \frac{\partial}{\partial\rho}\left(\frac{1}{2}n^2\right)\frac{\partial\rho}{\partial x} = \frac{x}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{2}n^2\right)$$

$$\bar{n}^2\frac{d^2y}{dz^2} = \cdots = \frac{y}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{2}n^2\right)$$

$$\bar{n}^2\frac{d^2x}{dz^2}\frac{dx}{dz} + \bar{n}^2\frac{d^2y}{dz^2}\cdot\frac{dy}{dz} = \frac{d}{dz}\left(\frac{1}{2}\bar{n}^2\left(\frac{dx}{dz}\right)^2 + \frac{1}{2}\bar{n}^2\left(\frac{dy}{dz}\right)^2\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}n^2\right)\frac{dx}{dz} + \frac{\partial}{\partial y}\left(\frac{1}{2}n^2\right)\frac{dy}{dz} = \frac{d}{dz}\left(\frac{1}{2}n^2\right)$$

$$\frac{d}{dz}\left[\frac{1}{2}\bar{n}^2\left(\frac{d\vec{x}}{dz}\right)^2 - \frac{1}{2}n^2\right] = 0 \qquad \frac{1}{2}\bar{n}^2\left(\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2\right) - \frac{1}{2}n^2 = -\frac{1}{2}\bar{n}^2$$

Energy

$$\frac{d}{dz}\left(x\frac{dy}{dz} - y\frac{dx}{dz}\right) = \cancel{\frac{dx}{dz}\frac{dy}{dz}} + x\frac{d^2y}{dz^2} - \cancel{\frac{dy}{dz}\frac{dx}{dz}} - y\frac{d^2x}{dz^2}$$

$$= x\left(\frac{1}{\bar{n}^2}\frac{y}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{2}n^2\right)\right) - y\left(\frac{1}{\bar{n}^2}\frac{x}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{2}n^2\right)\right) = 0$$

Angular momentum