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Magnetohydrodynamics (MHD) §7.7

given $\vec{E}, \vec{B} \longrightarrow \vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \rightarrow \rho\vec{E} + \vec{J}\times\vec{B}$

given $\rho, \vec{J} \longrightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad \vec{\nabla}\times\vec{B} = \mu\vec{J}$ ( )

In general - each affects the other.

Dynamics of continuous $\rho_m, \vec{v}$ Hydrodynamics

(Add $\vec{B}$. MHD) (Add $\rho_e, \frac{\partial\vec{E}}{\partial t}$ plasma physics)

Hydrodynamics:

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$$

continuity.
conservation of $M = \int d^3x\, \rho$.

Euler

Navier Stokes

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p - \rho_g\vec{\nabla}\Phi_g - \rho_e\vec{\nabla}\Phi_e + \vec{J}\times\vec{B} + \cdots$$
$$+ \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\vec{\nabla}\cdot\vec{v})\right)\right]$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} = \text{"convective derivative"}$$
(along flow)

②

Static background $\rho = \rho_0 \quad \vec{v} = 0.$

perturb $\rho = \rho_0 + \delta\rho \quad \vec{v} = 0 + \delta\vec{v} \qquad \delta$. "small"

$$\frac{\partial}{\partial t}(\cancel{\rho_0} + \delta\rho) + \vec{\nabla}\cdot\left[(\rho_0 + \cancel{\delta\rho})(\delta\vec{v})\right] = 0$$

$$\boxed{\frac{\partial}{\partial t}(\delta\rho) + \rho_0 \vec{\nabla}\cdot(\delta\vec{v}) = 0}$$

$$(\rho_0 + \cancel{\delta\rho})\left(\frac{\partial}{\partial t}(\delta\vec{v}) + \cancel{(\delta\vec{v}\cdot\vec{\nabla})\delta\vec{v}}\right) = -\vec{\nabla} p$$

Equation of state : $p = p(\rho)$

$$p(\rho_0 + \delta\rho) = p(\rho_0) + \left(\frac{\partial p}{\partial \rho}\right)_0 \cdot \delta\rho + \cdots$$

$$\boxed{\rho_0 \frac{\partial}{\partial t}(\delta\vec{v}) = -\left(\frac{\partial p}{\partial \rho}\right)_0 \cdot \vec{\nabla}\delta\rho}$$

$$\frac{\partial^2}{\partial t^2}(\delta\rho) = -\rho_0 \frac{\partial}{\partial t}(\vec{\nabla}\cdot\delta\vec{v})$$

$$\rho_0 \vec{\nabla}\cdot\left(\frac{\partial}{\partial t}(\delta\vec{v})\right) = -\left(\frac{\partial p}{\partial \rho}\right)_0 \nabla^2(\delta\rho)$$

$$\boxed{\nabla^2(\delta\rho) - \frac{1}{(\partial p/\partial\rho)_0}\frac{\partial^2}{\partial t^2}(\delta\rho) = \nabla^2(\delta\rho) - \frac{1}{c_s^2}\frac{\partial^2(\delta\rho)}{\partial t^2} = 0}$$

Wave equation : $\boxed{c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_0}$

ideal gas $\quad p = \rho \frac{kT}{m}$

isothermal $\quad \frac{\partial p}{\partial \rho} = \frac{kT}{m}$

adiabatic $\quad pV^{\gamma} = \text{constant}$

$p = \text{constant} \cdot \rho^{\gamma}$ $\quad \boxed{\frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho} = \frac{\gamma kT}{m}}$

N_2, O_2 diatomic $\quad \gamma = \frac{7}{5}$

$$m = (0.8)(28) + (0.2)(32) = 28.8\ m_p.$$

$$kT = (1\,eV)\left(\frac{T}{11{,}600\,K}\right) \simeq \frac{1}{40}\,eV \quad (17\,C = 62\,F)$$

$$c_s^2 = \sqrt{\frac{(1.4)\left(\frac{1}{40}\,eV\right)}{(28.8)(0.938 \times 10^{9}\,eV/c^2)}}$$

$$= \sqrt{1.295 \times 10^{-12}}\ c = 1.138 \times 10^{-6}\ c = 340\ m/s$$

$$\boxed{\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p = -\left(\frac{\partial p}{\partial \rho}\right)_0 \vec{\nabla} \rho}$$

$$\rho = \rho_0 + \rho_1 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{v} = \vec{v}_1 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\cancel{-i}\omega \rho_0 \vec{v}_1 = -c_s^2 \cancel{i}\vec{k} \rho_1 \qquad \rho_0 \vec{v}_1 = \frac{c_s^2 \vec{k}}{\omega} \rho_1 \qquad \omega^2 = c_s^2 k^2$$

$$\boxed{\vec{v}_1 = c_s \hat{k} \left(\frac{\rho_1}{\rho_0}\right)}$$

Add $\vec{E}, \vec{B}$ maxwell + MHD.

$$\vec{\nabla}\cdot\vec{E} = \rho_e/\varepsilon_0 \simeq 0$$ (MHD definition).

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$\vec{E}$ not "dynamic"

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\varepsilon \underbrace{\frac{\partial \vec{E}}{\partial t}}_{\to 0 \text{ (MHD)}} = \mu\sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$\vec{J} = \sigma\vec{E}'$

$$\Rightarrow \boxed{\vec{E} = \frac{1}{\mu\sigma}\left(\vec{\nabla}\times\vec{B}\right) - \vec{v}\times\vec{B}}$$ (constrained).

$$\vec{\nabla}\times\vec{E} = \frac{1}{\mu\sigma}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) - \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)$$

$$= \frac{1}{\mu\sigma}\left(\cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{B})} - \nabla^2\vec{B}\right) - \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) = -\frac{\partial \vec{B}}{\partial t}$$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) + \frac{1}{\mu\sigma}\nabla^2\vec{B}}$$ (7.68).

to scan

Hd

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0}$$

$$\boxed{\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p + \vec{J}\times\vec{B}}$$

$$\vec{J} = \frac{1}{\mu}\left(\vec{\nabla}\times\vec{B}\right) \qquad \vec{J}\times\vec{B} = -\frac{1}{\mu}\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)$$

Aside: magnetic pressure:

$$\vec{B}\times(\vec{\nabla}\times\vec{B}) = B_j(\nabla_i B_j) - B_j(\nabla_j B_i)$$

$$= \vec{\nabla}\left(\tfrac{1}{2}B^2\right) - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p - \vec{\nabla}\left(\frac{B^2}{2\mu}\right) + \frac{1}{\mu}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

(JDJ old edition, eq. following (7.69)) $\cancel{\mu}$ → $\frac{1}{\mu}$

$\frac{B^2}{2\mu}$ acts something like a pressure

Scenario $\rho = \rho_0 + \delta\rho$ $\vec{v} = 0 + \delta\vec{v}$ $\vec{B} = \vec{B}_0 + \delta\vec{B}$

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0\,\vec{\nabla}\cdot(\delta\vec{v}) = 0$$

$$\rho_0\frac{\partial}{\partial t}(\delta\vec{v}) = -c_s^2\,\vec{\nabla}(\delta\rho) - \frac{1}{\mu}\vec{B}\times(\vec{\nabla}\times\delta\vec{B})$$

$$\frac{\partial}{\partial t}(\delta\vec{B}) = \vec{\nabla}\times(\delta\vec{v}\times\vec{B}_0)$$

scalar + 2 vector $\delta\rho$, $\delta\vec{v}$, $\delta\vec{B}$