1/27/2015.

MHD $\quad \rho = \rho_0 + \delta\rho \quad \vec{v} = 0 + \delta\vec{v} \quad \vec{B} = \vec{B}_0 + \delta\vec{B}$

Continuity ①

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0 \vec{\nabla}\cdot(\delta\vec{v}) = 0$$

$$\vec{J} = \vec{\nabla}\times\vec{B}$$

Euler: ②

$$\rho_0 \frac{\partial}{\partial t}(\delta\vec{v}) = -c_s^2 \vec{\nabla}(\delta\rho) + \frac{1}{\mu}\vec{B}_0\times(\vec{\nabla}\times\delta\vec{B})$$

Faraday ③

$$\vec{E} + \vec{v}\times\vec{B} = 0$$

$$\frac{\partial}{\partial t}(\delta\vec{B}) = \vec{\nabla}\times(\delta\vec{v}\times\vec{B}_0)$$

Master equation

$\vec{\nabla}$ ①
$$\frac{\partial}{\partial t}(\vec{\nabla}\,\delta\rho) = -\rho_0\,\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v})$$

$\frac{\partial}{\partial t}$ ②
$$\rho_0\frac{\partial^2}{\partial t^2}(\delta\vec{v}) = -c_s^2\,\frac{\partial}{\partial t}(\vec{\nabla}\,\delta\rho) - \frac{1}{\mu}\vec{B}_0\times\left(\vec{\nabla}\times\frac{\partial}{\partial t}\delta\vec{B}\right)$$

③
$$\frac{\partial}{\partial t}(\delta\vec{B}) = \vec{\nabla}\times(\delta\vec{v}\times\vec{B}_0)$$

$$\Rightarrow\quad \rho_0\frac{\partial^2}{\partial t^2}(\delta\vec{v}) = +c_s^2\,\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v})\,\rho_0 - \frac{1}{\mu}\vec{B}_0\times\left[\vec{\nabla}\times(\vec{\nabla}\times(\delta\vec{v}\times\vec{B}_0))\right]$$

Let $\vec{V}_A = \dfrac{\vec{B}_0}{\sqrt{\mu\rho_0}}$ "Alfvén Velocity"

Hannes Alfvén, Swedish (1970)

$$\boxed{\frac{\partial^2}{\partial t^2}(\delta\vec{v}) = c_s^2\,\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v}) - \vec{V}_A\times\left[\vec{\nabla}\times(\vec{\nabla}\times(\delta\vec{v}\times\vec{V}_A))\right]}$$

(7.72)

(Parks ✱)

Let $\delta\vec{v} = \vec{v}_1\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$-\omega^2\,\vec{v}_1 = [\text{Matrix}]\,\vec{v}_1$$

eigenvalue problem $(\omega, \vec{v}_1)$

③

$$-\omega^2 \vec{v}_1 + c_s^2 \vec{k}(\vec{k}\cdot\vec{v}_1)$$

$$- \vec{v}_A \times [\vec{k}\times(\underbrace{\vec{k}\times(\vec{v}_1\times\vec{v}_A)}_{\vec{v}_1(\vec{k}\cdot\vec{v}_A)-\vec{v}_A(\vec{k}\cdot\vec{v}_1)})] = 0.$$

$$\rightarrow \quad -(\vec{k}\cdot\vec{v}_A)\,[\vec{k}(\underset{②}{\vec{v}_A\cdot\vec{v}_1}) - \vec{v}_1(\underset{①}{\vec{k}\cdot\vec{v}_A})]$$

$$+ (\vec{k}\cdot\vec{v}_1)\,[\vec{k}(\underline{\vec{v}_A\cdot\vec{v}_A}) - \vec{v}_A(\underset{③}{\vec{k}\cdot\vec{v}_A})]$$

$$-\omega^2 \vec{v}_1 + (c_s^2 + v_A^2)\,\vec{k}(\vec{k}\cdot\vec{v}_1) \qquad (7.75)$$

$$+ \underset{①}{(\vec{k}\cdot\vec{v}_A)^2\,\vec{v}_1} - \vec{k}(\underset{②}{\vec{k}\cdot\vec{v}_A})(\vec{v}_A\cdot\vec{v}_1)$$

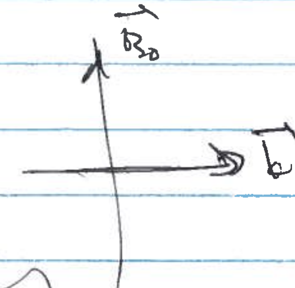
$$- \vec{v}_A(\underset{③}{\vec{k}\cdot\vec{v}_A})(\vec{k}\cdot\vec{v}_1) = 0$$

Structure: $[\text{matrix}]\,\vec{v}_1 = 0$

matrix invertible $\rightarrow$ $M^{-1}M\vec{v} = M^{-1}0 = \boxed{0 = \vec{v}}$

so: not invertible, $\boxed{\det M = 0}$ HW.

Special Cases:



I) $\vec k \perp \vec B_0$, $\vec k \perp \vec v_A$, $\vec k\cdot\vec v_A = 0$

$$-\omega^2 \vec v_1 + (c_s^2+v_A^2)\vec k(\vec k\cdot\vec v_1) = 0$$

In general: $\vec v_1 = \hat k(\hat k\cdot\vec v_1) + [\vec v_1 - \hat k(\hat k\cdot\vec v_1)] = \vec v_{1\parallel} + \vec v_{1\perp}$

$$-\omega^2(\vec v_\parallel + \vec v_\perp) + c_s^2 k^2 \vec v_\parallel = 0$$

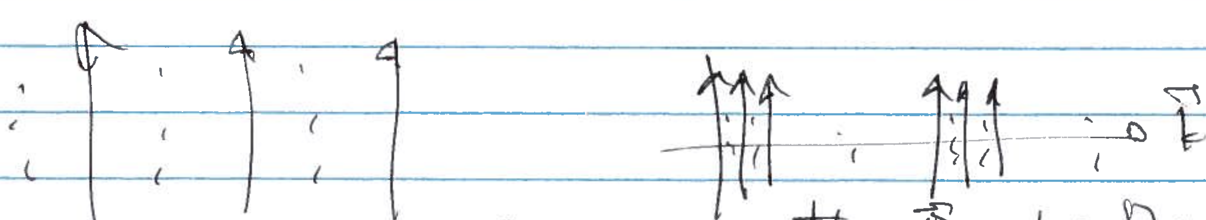
$\vec v_\perp = 0$ $\qquad -\omega^2 + (c_s^2+v_A^2)k^2 = 0.$

$\vec v_1 \parallel \vec k$ $\qquad \omega^2 = (c_s^2+v_A^2)k^2$

$-i\omega\rho_1 + \rho_0(i\vec k\cdot\vec v_1) = 0$ $\qquad \rho_1 = \rho_0 \dfrac{\vec k\cdot\vec v_1}{\omega} = \rho_0 \dfrac{\hat k\cdot\vec v_1}{\sqrt{c_s^2+v_A^2}}$

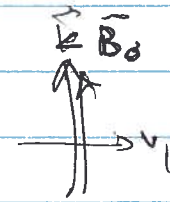
$$-i\omega\vec B_1 = i\vec k\times(\vec v_1\times\vec B_0) = i[\vec v_1\overbrace{(\vec k\cdot\vec B_0)}^{0} - \vec B_0(\vec k\cdot\vec v_1)]$$

$\vec B_1 = \vec B_0 \dfrac{\hat k\cdot\vec v_1}{\sqrt{c_s^2+v_A^2}}$ $\qquad$ "longitudinal magnetosonic wave"



squeezes matter, $\vec B$ together

$\vec{k} \parallel \vec{B}_0 \; (\vec{v}_A)$ $\qquad \vec{k}(\vec{v}_A \cdot \vec{v}_1) = v_A(\vec{k} \cdot \vec{v}_1)$



$$-\omega^2 + \vec{v}_1 \cancel{(c_s^2 + v_A^2)} \frac{k^3}{v_A^2} \vec{v}_A(\vec{v}_A \cdot \vec{v}_1)$$

$$+ k^2 v_A^2 \vec{v}_1 - k^2 \vec{v}_A(\vec{v}_A \cdot \vec{v}_1) - \cancel{\vec{v}_A k^2 (\vec{v}_A \cdot \vec{v}_1)} = 0$$

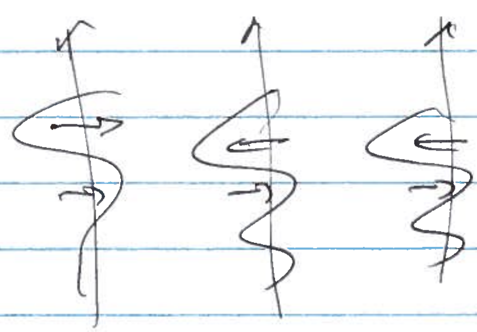
$$-\omega^2 \vec{v}_1 + k^2 v_A^2 \vec{v}_1 + \left(\frac{c_s^2}{v_A^2} - 1\right) k^2 (\vec{v}_A \cdot \vec{v}_1)\vec{v}_A = 0.$$

(A). $\vec{v}_A \cdot \vec{v}_1 = 0$ $\qquad \boxed{\omega^2 = v_A^2 k^2}$

$$-i\omega\rho_1 + \rho_0(i\cancel{\vec{k} \cdot \vec{v}_1}) = 0 \qquad \rho_1 = 0$$

$$-i\omega\vec{B}_1 = i\vec{k} \times (\vec{v}_1 \times \vec{B}_0) = i\left[\vec{v}_1(\vec{k} \cdot \vec{B}_0) - \cancel{\vec{B}_0(\vec{k} \cdot \vec{v}_1)}\right]$$

$$\vec{B}_1 = -\vec{v}_1\left(\frac{\vec{k} \cdot \vec{B}_0}{\omega}\right) = -\frac{\vec{v}_1}{v_A}\left(\frac{\vec{v}_A \cdot \vec{B}_0}{v_A}\right) = -\frac{\vec{v}_1}{v_A}\vec{B}_0$$



No density fluctuation

("pluck" $\vec{B}$)

Alfvén wave