2/1/2016 Chapter 9 Radiating Systems

oscillating sources. $\rho, \vec{J} \sim e^{-i\omega t}$ [Re]

Lorentz gauge $\boxed{\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0}$ (μ_0, ε_0)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\rho/\varepsilon_0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

$e^{-i\omega t}$

$k = \frac{\omega}{c}$

$$\longrightarrow \boxed{\begin{aligned}(\nabla^2 + k^2)\Phi &= -\rho/\varepsilon_0 \\ (\nabla^2 + k^2)\vec{A} &= -\mu_0\vec{J}\end{aligned}}$$

"Helmholtz Equation"

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi = \frac{\hbar^2 k^2}{2m}\psi$$

$$\longrightarrow \boxed{(\nabla^2 + k^2)\psi = \frac{2mV}{\hbar^2}\psi}$$

Green's Function

$$G(\vec{x}, \vec{x}') = \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

(Sign ⇔ B.C. outgoing waves)

two ways

① $$(\nabla^2+k^2)\frac{e^{ikR}}{R} = e^{ikR}\left(\nabla^2\frac{1}{R}\right) + 2\vec{\nabla}(e^{ikR})\cdot\vec{\nabla}\left(\frac{1}{R}\right) + \frac{1}{R}\nabla^2\left[e^{ikR}\right] + k^2\cdot\frac{e^{ikR}}{R}$$

$$= e^{ikR}\left(-4\pi\,\delta(\vec{R})\right) + 2(ik\hat{R})e^{ikR}\cdot\left(-\frac{\hat{R}}{R^2}\right) + \frac{1}{R}\left(\cancel{-k^2} + \frac{2(ik)}{R}\right)e^{ikR} + \cancel{k^2\frac{e^{ikR}}{R}}$$

$$= e^{ikR}\left(-4\pi\,\delta(\vec{R})\right) = -4\pi\,\delta(\vec{R})$$

② $$G(\vec{x}, t; \vec{x}', t') = \frac{1}{|\vec{x}-\vec{x}'|}\delta\left(t-t'-\frac{1}{c}R\right)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

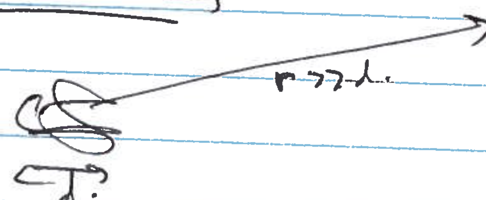
$$\vec{A} = \int d^3x'\,dt'\,\frac{\mu_0}{4\pi}\,\vec{J}(\vec{x}')e^{-i\omega t'}\frac{1}{|\vec{x}-\vec{x}'|}\delta\left(t'-t-\frac{1}{c}R\right)$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{J}(\vec{x}')\,e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}e^{-i\omega t}}{|\vec{x}-\vec{x}'|}$$

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}(\vec{x}')\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}} \; e^{-i\omega t}$$

- perturbation theory $kd \ll 1$.
- exact solution
- multipole series $\sum \vec{X}_{lm}$.

all start from localized sources



$$R = |\vec{x}-\vec{x}'| = \sqrt{r^2 + r'^2 - 2\vec{x}\cdot\vec{x}'}$$

$$= r\left(1 - \frac{2\vec{x}\cdot\vec{x}'}{r^2} + \frac{r'^2}{r^2}\right)^{\frac{1}{2}} \simeq r\left(1 - \frac{\vec{x}\cdot\vec{x}'}{r^2}\right) + \cdots$$

$$\boxed{R \simeq r - \hat{r}\cdot\vec{x}'}$$

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \, \vec{J}(\vec{x}')\, e^{-ik\hat{r}\cdot\vec{x}'}} \qquad (9.8) \; (\ast\ast\ast)$$

$$\vec{A} = \frac{e^{ikr}}{r}\vec{F}(\hat{r})$$

radiation regime - always

$$\vec{B} = \vec{\nabla}\times\vec{A} = \left(\hat{r}\frac{\partial}{\partial r} + \cancel{\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}} + \cancel{\frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}}\right)\times\vec{A}$$

$$= ik\hat{r}\times\vec{F}\,\frac{e^{ikr}}{r}$$

$$\frac{\partial}{\partial r}\left(e^{ikr}\right) = ik$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}$$

$$\vec{\nabla}\times\vec{B} = \cancel{ik}\,\hat{r}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = -\frac{i\omega}{c^2}\vec{E} = \frac{\cancel{ik}}{c}\vec{E}$$

$$\vec{E} = -ick\,\hat{r}\times(\hat{r}\times\vec{F})\,\frac{e^{ikr}}{r}$$

$$= +i\omega\,(\hat{r}\times\vec{F})\times\hat{r}\,\frac{e^{ikr}}{r}$$

$$\vec{F}(\hat{r}\cdot\hat{r}) - \hat{r}(\hat{r}\cdot\vec{F}) = \vec{F}_\perp$$

Both $\vec{E}$, $\vec{B}$ $\perp$ to $\hat{r}$

$$\langle\vec{S}\rangle = Re\left(\frac{1}{2}\vec{E}\times\vec{H}^*\right)$$

$$= \frac{1}{2\mu_0} Re\left[i\omega\,(\hat{r}\times\vec{F})\times\hat{r}\,\frac{e^{ikr}}{r}\times\left(ik\hat{r}\times\vec{F}\,\frac{e^{ikr}}{r}\right)^*\right]$$

$$= \frac{1}{2\mu_0}\frac{\omega k}{r^2}\left[\left((\hat{r}\times\vec{F})\times\hat{r}\right)\times(\hat{r}\times\vec{F})^*\right]$$

$$= \frac{1}{2\mu_0}\frac{ck^2}{r^2}\left[\hat{r}\,|\hat{r}\times\vec{F}|^2 - (\hat{r}\times\vec{F})\cancel{\left[\hat{r}\cdot(\hat{r}\times\vec{F})^*\right]}\right]$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = r^2\,\hat{r}\cdot\langle\vec{S}\rangle = \frac{1}{2\mu_0}ck^2\,|\hat{r}\times\vec{F}|^2$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{2\mu_0}ck^2\,|(\hat{r}\times\vec{F})\times\hat{r}|^2$$

$$\propto |\vec{E}|^2$$

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$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}')\,e^{-ik\hat{r}\cdot\vec{x}'}}$$

exact for $r \gg d$.
"radiation regime"

$k\hat{r}$ = "$\vec{k}_{out}$", F.T. of $\vec{J}(\vec{x}')$.

~~long~~ "Long wavelength" $kd \ll 1$. $d \ll \lambda$.

$$e^{-ik\hat{r}\cdot\vec{x}'} \simeq 1 - ik\hat{r}\cdot\vec{x}' + \dots$$

leading.

$$\cancel{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int} \quad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}')$$

Static $\int d^3x'\,\vec{J}(\vec{x}') = 0$. not any more.

$$\nabla'_j(x'_i J_j) = \delta_{ij}\cdot J_j + x'_i(\nabla'_j J_j) = \vec{J} + \vec{x}'(\vec{\nabla}\cdot\vec{J})$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = -i\omega\rho + \vec{\nabla}\cdot\vec{J} \qquad \boxed{\vec{\nabla}\cdot\vec{J} = i\omega\rho}$$

$$\int d^3x'\,\vec{J}(\vec{x}') = \int d^3x'\left[\nabla'_j(x'_i J_j) - \vec{x}'(\vec{\nabla}\cdot\vec{J})\right]$$

$\hookrightarrow \oint d^2a\,n_j \dots \to 0$.

$$= -\int d^3x'\,i\omega\rho\,\vec{x}' = -i\omega\vec{p}$$

Electric dipole moment constant