2/3/2016

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} \rightarrow \left(\nabla^2 + k^2\right)\vec{A} = -\mu_0 \vec{J} \qquad \frac{\partial}{\partial t} \to -i\omega$$

Maxwell's equations

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}(\vec{x}') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \qquad e^{-i\omega t}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \, \vec{J}(\vec{x}') e^{-ik\hat{r}\cdot\vec{x}'} \qquad r \gg d.$$

$kd \ll 1$ ~~long wavelength~~ long wavelength $\qquad e^{-ik\hat{r}\cdot\vec{x}'} \approx 1$

$$\int d^3x' \, \vec{J}(\vec{x}') = \int d^3x' \left[-\vec{x}'(\vec{\nabla}\cdot\vec{J})\right]$$

$$= -i\omega \int d^3x' \, \vec{x}' \rho(\vec{x}') = -i\omega \vec{p}.$$

$$\vec{A}_{E1} = -i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{p} \qquad (9.16)$$

Charge conserved. — can't have oscillating Q.

GR. mass, momentum conserved → quadrupole

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = \frac{1}{\mu_0} \left(\hat{r} \frac{\partial}{\partial r} \right) \times \left(-i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{p} \right)$$

$$= -\frac{i\omega}{4\pi} \left(ik - \frac{1}{r} \right) \hat{r} \times \vec{p} \, \frac{e^{ikr}}{r}$$

$$\vec{H} = \frac{ck^2}{4\pi} (\hat{r} \times \vec{p}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right)$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega \varepsilon_0 \vec{E} = -ick\varepsilon_0 \vec{E}$$

(9.20)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \left[k^2 (\hat{r} \times \vec{p}) \times \hat{r} \, \frac{e^{ikr}}{r} + \frac{3\hat{r}(\hat{r} \cdot \vec{p}) - \vec{p}}{r^3} e^{ikr} (1 - ikr) \right]$$

$kr \to 0$ $(kr \to 0)$ $\vec{H} \to 0$ (k^2)

$\vec{E} \to$ (static dipole field) $\times e^{-i\omega t}$

$kr \to \infty$ $\vec{E}, \vec{H} \sim \frac{e^{ikr}}{r}$ radiation fields.

$kr \sim 1$. intermediate (transition) all terms

power

$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^*$ (9.19)

$= \frac{1}{2} \left(\frac{1}{4\pi\varepsilon_0} k^2 (\hat{r} \times \vec{p}) \times \hat{r} \frac{e^{ikr}}{r} \right) \times \left(\frac{c k^2}{4\pi} (\hat{r} \times \vec{p}) \frac{e^{ikr}}{r} \right)^*$

$= \frac{1}{32\pi^2 \varepsilon_0} \frac{ck^2}{r^2} \left[\left((\hat{r} \times \vec{p}) \times \hat{r} \right) \times \left(\hat{r} \times \vec{p} \right)^* \right]$

$\hat{r} |\hat{r} \times \vec{p}|^2$

$$\frac{dP}{d\Omega} = \frac{ck^2}{32\pi^2 \varepsilon_0} \left| (\hat{r} \times \vec{p}) \times \hat{r} \right|^2$$

radial inverse sq. rule.

insert $c\sqrt{\mu_0 \varepsilon_0}$

$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.9\,\Omega$

$$\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} Z_0 k^4 \left| (\hat{r} \times \vec{p}) \times \hat{r} \right|^2$$

(9.22)

$\hat{n} = \hat{r}$

angular distribution

(4)

$$|\hat{z}\times\hat{r}|^2 = |\hat{z}\times(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)|^2$$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi = \sin^2\theta$$

$$|(\hat{x}+i\hat{y})\times\hat{r}|^2 = |\hat{x}\times(\sin\theta\sin\phi\,\hat{y} + \cos\theta\hat{z}) + i\,\hat{y}\times(\sin\theta\cos\phi\,\hat{x} + \cos\theta\hat{z})|^2$$

$$= |e^{i\phi}\sin\theta(\sin\phi + i\cos\phi)\hat{z} + i\cos\theta(\hat{x}+i\hat{y})|^2$$

$$= \sin^2\theta + \cos^2\theta + \cos^2\theta = 1+\cos^2\theta$$

$\frac{1}{2}$

$$P = \int d\Omega\,\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2}\int d\Omega\,|\hat{r}\times\vec{p}|^2$$

$$|\hat{r}\times\vec{p}|^2 = (\hat{r}\times\vec{p})\cdot(\hat{r}\times\vec{p}^*) = \hat{r}\cdot(\vec{p}\times(\hat{r}\times\vec{p}^*))$$

$$= \hat{r}\cdot(\hat{r}(\vec{p}\cdot\vec{p}^*) - \vec{p}^*(\hat{r}\cdot\vec{p})) = \vec{p}\cdot\vec{p}^* - (\hat{r}\cdot\vec{p})(\hat{r}\cdot\vec{p}^*)$$

$$P = \frac{c^2 Z_0 k^4}{32\pi^2}\int d\Omega\left[\underbrace{\vec{p}\cdot\vec{p}^*}_{4\pi} - \underbrace{(\hat{r}_i\hat{r}_j)p_i p_j^*}_{\frac{4\pi}{3}\delta_{ij}}\right]$$

$$P = \frac{c^2 Z_0 k^4}{32\pi^2}|\vec{p}|^2\left(4\pi - \frac{4\pi}{3}\right) = \frac{c^2 Z_0}{12\pi}|\vec{p}|^2$$

$$cZ_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\varepsilon_0} \rightarrow \frac{1}{3}\frac{ck^4}{(4\pi\varepsilon_0)}|\vec{p}|^2$$

$e^{-ik\hat{r}\cdot\vec{x}'} = 1 - ik\hat{r}\cdot\vec{x}' + \dots$ next order term.

$$(\hat{r}\cdot\vec{x}')\vec{J} = \tfrac{1}{2}\left[(\hat{r}\cdot\vec{x}')\vec{J} + (\hat{r}\cdot\vec{J})\vec{x}'\right] + \tfrac{1}{2}\left[(\hat{r}\cdot\vec{x}')\vec{J} - (\hat{r}\cdot\vec{J})\vec{x}'\right]$$

$$(\vec{x}'\times\vec{J})\times\hat{r} = \vec{J}(\vec{x}'\cdot\hat{r}) - \vec{x}'(\vec{J}\cdot\hat{r})$$

Asymmetric piece:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\int d^3x' \tfrac{1}{2}(\vec{x}'\times\vec{J})\right]\times(-ik\hat{r})$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, ik\,\hat{r}\times\vec{m}}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{ik\hat{r}}{\mu_0}\times\vec{A} = -\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\hat{r}\times(\hat{r}\times\vec{m})$$

$$\boxed{\vec{H} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{r}\times\vec{m})\times\hat{r}}$$

$$\vec{\nabla}\times\vec{H} = ik\hat{r}\times\vec{H} = ik\left(\frac{k^2}{4\pi}\right)\frac{e^{ikr}}{r}\hat{r}\times\vec{m} = -i\omega\varepsilon_0\vec{E}$$

$$\boxed{\vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}\hat{r}\times\vec{m}}$$

Electric, magnetic dipoles : duality.

$\vec{E} \to Z_0 \vec{H}$ $\quad \vec{E} \to c\vec{B}$

$Z_0 \vec{H} \to -\vec{E}$ $\quad c\vec{B} \to -\vec{E}$

$\vec{p} \to \frac{\vec{m}}{c}$

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} Z_0 k^4 \left| \hat{r} \times \vec{m} \right|^2$$

$$P = \frac{c^2 Z_0 k^4}{12\pi} \left| \frac{\vec{m}}{c} \right|^2$$

Symmetric part :

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \int d^3x' \frac{1}{2} \left[(\hat{r} \cdot \vec{x}') \vec{J} + (\hat{r} \cdot \vec{J}) \vec{x}' \right]$$

$$\hat{r}_j \left(x'_i J_j + x'_j J_i \right)$$

$$\nabla'_k \left(x'_i x'_j J_k \right) = \delta_{ik} x'_j J_k + x'_i \delta_{jk} J_k + x'_i x'_j \nabla_k J_k$$

$$= x'_i J_j + x'_j J_i + i\omega \rho \, x'_i x'_j$$

$$\int d^3x \, \nabla_k (\cdot) \Rightarrow \oint d\vec{s}' \, \hat{n}_k \to 0$$

$$\int d^3x' \left(x'_i J_j + x'_j J_i + i\omega \rho \, x'_i x'_j \right) = 0$$