2/8/2016

Electric dipole $\vec{A}_{ED} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-i\omega \vec{p})$

magnetic dipole $\vec{A}_{MD} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (ik \hat{r} \times \vec{m})$

Electric quadrupole $\vec{A}_{E.Q.} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(-\frac{ck^2}{6} \vec{Q}(\hat{r}) \right)$

$$P = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2$$

$$P = \frac{c^2 Z_0 k^4}{12\pi} \left| \frac{\vec{m}}{c} \right|^2$$

$$P = \frac{c^2 Z_0 k^6}{1440\pi} |Q_{ij}|^2 \qquad (Q^*_{ij} Q_{ij} \ .)$$

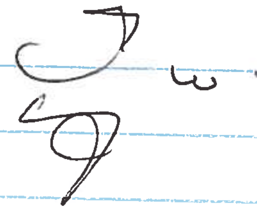
9.49

axially symmetric $Q_{ij} = \begin{pmatrix} -\frac{1}{2}Q_0 & & \\ & -\frac{1}{2}Q_0 & \\ & & Q_0 \end{pmatrix}$

$$Q^*_{ij} Q_{ij} = |Q_0|^2 \left(\frac{1}{4} + \frac{1}{4} + 1 \right) = \frac{3}{2} |Q_0|^2$$

$$P = \frac{c^2 Z_0 k^4 |Q_0|^2}{960\pi} \qquad (9.52)$$

Rotating :



periodic $\rightarrow \vec{J} = \sum_{n=0}^{\infty} \vec{J}_n(\vec{x})\, e^{-in\omega t}$

$$\vec{J}_n = \frac{1}{T}\int_0^T dt\, \rho(\vec{x},t)\, e^{+in\omega t},$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x'\, dt'\, \vec{J}(\vec{x}',t')\, \frac{1}{|\vec{x}-\vec{x}'|}\, \delta\left(t'-t+\frac{1}{c}|\vec{x}-\vec{x}'|\right)$$

$$= \sum_{n=0}^{\infty} \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}_n(\vec{x}')\, \frac{e^{ink|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\, e^{-in\omega t}$$

$r \gg d$ $\rightarrow \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \int d^3x'\, \vec{J}_n(\vec{x}')\, e^{-ink\hat{r}\cdot\vec{x}'}$

$\rightarrow$ multipole moments $\vec{p}^{(n)}, \vec{m}^{(n)}, Q_{ij}^{(n)}$ ..

But:

$$Q_{ij} = \int d^3x\, (3x_ix_j - \delta_{ij}r^2)\, r^2 \left[\frac{1}{T}\int_0^T dt\, e^{+i\omega t} \rho(\vec{x},t)\right]$$

$$= \frac{1}{T}\int dt\, e^{+i\omega t}\left[\int d^3x\, (3x_ix_j - \delta_{ij}r^2)\,\rho\right]$$

$$\vec{x} = a\left(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}\right)$$

$$\vec{p} = q\vec{x} = qa\left(\hat{x}\cos\omega t + \hat{y}\sin\omega t\right)$$

$$= \mathrm{Re}\left[\cancel{qa}\,(\hat{x}+i\hat{y})\,e^{-i\omega t}\right)$$

$$\boxed{\vec{p}^{(1)} = qa(\hat{x}+i\hat{y})} \qquad \text{no } \vec{p}^{(n)} \quad n>1$$

$$\vec{m} = \int d^3x\, \tfrac{1}{2}(\vec{x}\times\vec{J}) = \tfrac{1}{2} q\vec{x}\times(\vec{\omega}\times\vec{x})$$

$$\vec{\omega}(\vec{x}\cdot\vec{x}) - \vec{x}(\vec{\omega}\cdot\vec{x})$$

$$\boxed{\vec{m}^{(0)} = \tfrac{1}{2}qa^2\vec{\omega}} \qquad \text{no other } \vec{m}^{(n)}$$

$$Q_{ijk} = \int d^3x\, \rho(\vec{x})\left[x_ix_jx_k - \frac{r^2}{5}\left(\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i\right)\right]$$

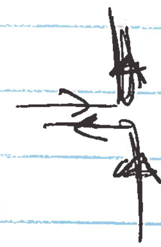
$\mathrm{Tr}\,Q \to 0$

$$Q_{111} = \begin{matrix} qx^3 \\ -3r^2x \end{matrix} = \cancel{qa^3 - 3} \quad = qa^3\left(\cos^3\omega t - \tfrac{3}{5}\cos\omega t\right)$$

$$= qa^3\left(\underbrace{\tfrac{1}{4}\cos 3\omega t}_{n=3} + \underbrace{\tfrac{3}{20}\cos\omega t}_{n=1}\right)$$

Systematic multipoles.

"Exact" toy model: linear antenna §9.4



$\vec{J}\, d^3x' \to \vec{I}\, dz'$

"centerfed" $I = I(|z|)$. symmetric.

Near: $I \sim \hat{z} \to \vec{A} = A\hat{z}$.

just outside. $(\nabla^2 + k^2)\vec{A} \to \left(\frac{\partial^2}{\partial z^2} + k^2\right)\vec{A} = 0$.

$A \sim e^{\pm ikz}$.

$I \sim e^{\pm ikz}$

$$I(z) = I_0 \cdot \sin\left[k\left(\frac{d}{2} - |z|\right)\right]$$

Idealized finite thickness. (take acc'd)

losses (radiative, ohmic) $e^{-k|z|}$

Continuity $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

$$\frac{\partial \lambda}{\partial t} + \frac{\partial I}{\partial z} = 0 \qquad i\omega\lambda = \frac{\partial I}{\partial z}$$

$$i\omega\lambda = \mp k I_0 \cos\left[k\left(\frac{d}{2} - |z|\right)\right] \qquad \lambda = \pm \frac{i I_0}{c} \ldots$$

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{r}\cdot\vec{x}'}$$

$$\longrightarrow \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} I_0 \hat{z} \cdot dz' \, \sin\left[k\left(\frac{d}{2} - |z'|\right)\right]$$

$$\times \, e^{-ikz'\cos\theta}$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I_0 \hat{z} \int_0^{d/2} dz' \, \sin\left[k\left(\frac{d}{2} - z'\right)\right]$$

$$\underbrace{\left(e^{-ikz'\cos\theta} + e^{+ikz'\cos\theta}\right)}_{2\cos(kz'\cos\theta)}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b)$$

$$= \sin\left(\frac{kd}{2} - kz' + kz'\cos\theta\right)$$

$$+ \sin\left(\frac{kd}{2} - kz' - kz'\cos\theta\right)$$

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{kr} 2I_0\hat{z} \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{1-\cos^2\theta}\right]$$

9.55

(common denominator)

$$\vec{H} = \frac{1}{\mu_0} ik\hat{r}\times\vec{A} \leftarrow (\sin\theta)$$

$$-i\omega\varepsilon_0\vec{E} = ik\hat{r}\times\vec{H}$$

$$\frac{dP}{d\Omega} = r^2\,\hat{r}\cdot\left(\frac{1}{2}\vec{E}\times\vec{H}^*\right) \qquad \frac{1}{\varepsilon_0 c} = \sqrt{\frac{\mu_0\varepsilon_0}{\varepsilon_0}} = \boxed{Z_0}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0|I_0|^2}{8\pi^2}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2}$$

(9.56).

poles $\sin^2\theta$, but near poles. $\theta \simeq \Delta\theta$, $\pi - \Delta\theta$

$$\cos\theta \simeq \left(1 - \frac{1}{2}\Delta\theta^2\right)$$

$$\cos\left(\frac{kd}{2}\cos\theta\right) \simeq \cos\left(\frac{kd}{2} - \frac{1}{2}\left(\frac{kd}{2}\right)\Delta\theta^2\right)$$

$$= \cos\frac{kd}{2}\cdot\underbrace{\cancel{\cos(\Delta\theta^2)}}_{1} - \sin\frac{kd}{2}\cdot\frac{1}{2}\left(\frac{kd}{2}\right)^2\Delta\theta^2$$

$$\frac{dP}{d\Omega} \to \frac{Z_0|I_0|^2}{8\pi^2}\left(\frac{\frac{1}{8}(kd)^2\Delta\theta^2}{\Delta\theta}\right)^2 \propto \Delta\theta^2 \to 0.$$

equator $\theta = \frac{\pi}{2}$ $\cos\theta = 0$. $\left[1 - \cos\left(\frac{kd}{2}\right)\right]^2$

max. $\cos\vartheta = -1$ $\frac{k\theta}{2} = \pi, + 2\pi n.$ $\boxed{kd = 2\pi(2n+1)}$

zero $\cos\vartheta = 1$ $\frac{kd}{2} = 2\pi n$ $\boxed{kd = 2n\cdot 2\pi}$

Case : $kd = \pi$ "half-wave"

$$\cos\frac{kd}{2} = \cos\frac{\pi}{2} = 0$$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0 |I_0|^2}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}$$

Case $kd = 2\pi$ "full-wave"

$$\cos\frac{kd}{2} = \cos\pi = -1$$

$$\frac{dP}{d\Omega} = \frac{Z_0 |I_0|^2}{8\pi^2} \frac{\left[1 + \cos(\pi\cos\theta)\right]^2}{\sin^2\theta}$$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0 |I_0|^2}{8\pi^2} \cdot 4 \cdot \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}$$