2/15/2016

$(\nabla^2+k^2)\vec{A}=0$ etc: could take $\vec{A}=\sum_{lm} A_{lm} h_l^{(1)} Y_l^{(1)}(k_0)$

works, but {need two polarizations
{need to build in $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ etc

In $\vec{E},\vec{H}$.

$\vec{\nabla}\cdot\vec{E}=0$ $\quad\vec{\nabla}\cdot\vec{H}=0$ (outside sources).

$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=+i\omega\vec{B}=i\omega\mu_0\vec{H}=ick\mu_0\vec{H}=ikZ_0\vec{H}$

$\vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}=-i\omega\varepsilon_0\vec{E}=-\frac{ik\vec{E}}{Z_0}$

$$\boxed{\vec{\nabla}\times\vec{E}=ik(Z_0\vec{H})}\qquad\boxed{\vec{\nabla}\times(Z_0\vec{H})=-ik\vec{E}}$$

will use as basis for spherical waves
$(\vec{r}\cdot\vec{E})$, $(\vec{r}\cdot\vec{H})$ $\quad$ ("$\vec{x}$" $\to$ "$\vec{r}$"). $\quad$ §9.7

Small Theorem:

$$\nabla^2(\vec{r}\cdot\vec{v})=\nabla_i\nabla_i(x_j v_j)=\nabla_i(\delta_{ij}v_j+x_j\nabla_i v_j)$$

$$=\nabla_i v_i+\delta_{ij}\nabla_i v_j+x_j\nabla^2 v_j$$

$$\boxed{\nabla^2(\vec{r}\cdot\vec{v})=2(\vec{\nabla}\cdot\vec{v})+\vec{r}\cdot(\nabla^2\vec{v})}$$

$(\nabla^2 + k^2)(\vec{r}\cdot\vec{E}) = 0$ $\qquad$ $(\nabla^2 + k^2)(\vec{r}\cdot\vec{H}) = 0$

$\vec{r}\cdot\vec{E} = \sum f_l Y_{lm}$ $\qquad$ $\vec{r}\cdot\vec{H} = \sum g_l Y_{lm}$

"electric multipole mode" $\qquad$ "magnetic multipole mode"

For full fields need to introduce

$$\boxed{\vec{L} = \frac{1}{i}(\vec{r}\times\vec{\nabla})}$$

($\hbar = 1$, $\vec{L}$ dimensionless)

$$\vec{L} = \frac{1}{i}(r\hat{r}) \times \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

$$\boxed{\vec{L} = \frac{1}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)}$$

lives in $\theta\,\phi$ space.
$\hat{\theta}, \hat{\phi}$ $\quad \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}$

$\hat{r}\cdot\vec{L} = 0$ $\qquad$ $\vec{L}f(r) = 0$ $\qquad$ $\frac{\partial \vec{L}}{\partial r} = 0$

$[\vec{L}, r] = 0$ $\qquad$ $[\vec{L}, \frac{\partial}{\partial r}] = 0$

$[\vec{L}, \vec{x}] \neq 0$ $\qquad$ $[\vec{L}, \hat{r}] \neq 0$.

$$\vec{r}\times\vec{L} = \vec{r}\times\left(\frac{1}{i}\vec{r}\times\vec{\nabla}\right) = \frac{1}{i}\left[\vec{r}(\vec{r}\cdot\vec{\nabla}) - r^2\vec{\nabla}\right]$$

$$= \frac{r^2}{i}\left[\hat{r}\frac{\partial}{\partial r} - \vec{\nabla}\right]$$

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} - \frac{i}{r}\hat{r}\times\vec{L}}$$

$L_z = -i\frac{\partial}{\partial\phi}$ $\quad L_z Y_{lm} = m Y_{lm}$ $\quad L^2 Y_{lm} = l(l+1)Y_{lm}$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ $\quad \hat{z}\cdot\vec{L} = (-\sin\theta)\left(+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right) = i\frac{\partial}{\partial\phi}$

"Electric multipole" (TM mode, $E_r \neq 0$)

$$\vec{r}\cdot\vec{E}^{E}_{lm} = -\frac{Z_0\, l(l+1)}{k} f_l(kr)\, Y_{lm}(\theta,\phi)$$

$(\vec{r}\cdot\vec{H}^{E}_{lm} = 0)$ $\quad$ ↑ $h^{(1)}_l(kr)$ for radiation

$$\vec{L}\cdot\vec{H}^{E}_{lm} = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{H} = \frac{1}{i}\vec{r}\cdot(\vec{\nabla}\times\vec{H})$$

$$= \frac{1}{i}\vec{r}\cdot\left(\frac{-ik\vec{E}}{Z_0}\right) = -\frac{k}{Z_0}(\vec{r}\cdot\vec{E})$$

$$= -\frac{k}{Z_0}\left(\left(-\frac{Z_0\, l(l+1)}{k}\right) f_l(kr)\, Y_{lm}\right)$$

$$\boxed{\vec{L}\cdot\vec{H} = l(l+1) f_l Y_{lm}} \quad \boxed{\vec{H}^{E}_{lm} = f_l(kr)\vec{L}Y_{lm}}$$

$$\boxed{E^{E}_{lm} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}^{E}_{lm}}$$

(4)

"magnetic multipole" (TE)

$$\vec{r}\cdot\vec{H}_{lm} = Z_0\, g_l(kr)\vec{L}\,Y_{lm} \qquad \vec{r}\cdot\vec{E}^{m}_{lm} = 0$$

$$\vec{E}^{m}_{lm} = Z_0\, g_l(kr)\vec{L}\,Y_{lm}$$

$$\vec{H}^{m}_{lm} = \frac{-i}{kZ_0}\vec{\nabla}\times\vec{E}^{m}_{lm}$$

$$\frac{\vec{L}\,Y_{lm}}{\sqrt{l(l+1)}} = \vec{X}_{lm}$$

vector spherical harmonic

$$\vec{H} = \sum_{l,m}\left[a^{E}_{lm} f_l(kr)\vec{X}_{lm} - \frac{i}{k}a^{m}_{lm}\vec{\nabla}\times\left(g_l(kr)\vec{X}_{lm}\right)\right]$$

$$\vec{E} = Z_0\sum_{lm}\left[\frac{i}{k}a^{E}_{lm}\vec{\nabla}\times\left(f_l\vec{X}_{lm}\right) + a^{m}_{lm}\, g_l\vec{X}_{lm}\right]$$

(9.122)

(5)

$$\vec{\nabla}\cdot\left(f\vec{X}\right) = \left(\vec{r}\,\frac{\partial}{\partial r} - \frac{i}{r}\,\vec{r}\times\vec{L}\right)\cdot\left(f\vec{X}\right)$$

$$= \cancel{\frac{\partial f}{\partial r}\left(\vec{r}\cdot\vec{X}\right)} - \frac{if}{r}\left(\vec{r}\times\vec{L}\right)\cdot\vec{X}$$

$$= \frac{-if}{r}\left(\vec{r}\times\vec{L}\right)\cdot\frac{\vec{L}\,Y_{lm}}{\sqrt{l(l+1)}} = \frac{-if}{r}\,\vec{r}\cdot\left(\vec{L}\times\vec{L}\right)Y_{lm} \qquad \vec{L}\times\vec{L} \to i\vec{L} \qquad = 0$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times f\vec{X}\right) = \vec{\nabla}\cancel{\left(\vec{\nabla}\cdot f\vec{X}\right)} - \nabla^2\left(f\vec{X}\right) = +k^2 f\vec{X}$$

$$[\vec{L}, \nabla^2] = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{H} = 0$$

$$\left(\nabla^2+k^2\right)\vec{E} = 0 \qquad \left(\nabla^2+k^2\right)\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = ikZ_0\vec{H} \qquad \vec{\nabla}\times\vec{H} = -\frac{ik}{Z_0}\vec{E}$$

$$\vec{r}\cdot\vec{H} = \sum_{lm}\left[a^E_{lm}\,f_l\,\cancel{\vec{r}\cdot\vec{X}_{lm}} - \frac{i}{k}a^M_{lm}\,\vec{r}\cdot\left(\vec{\nabla}\times g_l\vec{X}\right)\right]$$

$$= \sum_{lm}\left(\frac{-ia^M_{lm}}{k}\right)\left(\vec{r}\times\vec{\nabla}\right)\cdot g_l\frac{\vec{L}\,Y_{lm}}{\sqrt{l(l+1)}}$$

$$\boxed{\vec{r}\cdot\vec{H} = \sum_{lm}\left(\frac{1}{k}\right)a_{lm}\sqrt{l(l+1)}\;g_l\,Y_{lm}}$$

radial component relatively $\sim X_r$

$$\boxed{\int d\Omega \, Y^*_{lm}(\vec{r}\cdot\vec{H}) = \frac{\sqrt{l(l+1)}}{k} a^m_{lm} \, g_l(kr)}$$

$$\boxed{\int d\Omega \, Y^*_{lm}(\vec{r}\cdot\vec{E}) = -\frac{\sqrt{l(l+1)}}{k} Z_0 a^E_{lm} \, f_l(kr)}$$

$$\{a^m_{lm}, a^E_{lm}\} \longleftrightarrow (\vec{E}, \vec{H})$$

Outgoing waves $f_l, g_l = h^{(1)}_l(kr)$

$(kr \gg l)$ $\quad h^{(1)}_l \sim (-i)^{l+1} \frac{e^{ikr}}{kr}$

$$\vec{\nabla} \times \left((-i)^{l+1} \frac{e^{ikr}}{r} \vec{X}_{lm} \right) \Rightarrow ik\hat{r} \times (-i)^{l+1} \frac{e^{ikr}}{\cancel{k}r} \vec{X}_{lm}$$

$$\vec{H} \rightarrow \frac{e^{ikr}}{kr} \sum_{l,m} \left(a^E_{lm} (-i)^{l+1} \vec{X}_{lm} + a^m_{lm} (-i)^{l+1} \hat{r} \times \vec{X}_{lm} \right)$$

$$\vec{E} \rightarrow Z_0 \frac{e^{ikr}}{kr} \sum_{lm} \left((-i)^{l+1} (-a^E_{lm} \hat{r} \times \vec{X}_{lm} + a^m_{lm} \vec{X}_{lm}) \right)$$

$$\boxed{\vec{E} = -Z_0 \hat{r} \times \vec{H}} \qquad (Z_0 \vec{H} = \hat{r} \times \vec{E}) \qquad (\text{Duality})$$

$$\frac{dP}{d\Omega} = r^2 \, Re\left[\hat{r} \cdot \frac{1}{2}\left(\vec{E} \times \vec{H}^*\right)\right]$$

$$= r^2 Z_0 \, \vec{H} \cdot \vec{H}^*$$

$$\boxed{\begin{aligned} \frac{dP}{d\Omega} &= \frac{Z_0}{2k^2}\left|\sum_{l,m} (-i)^{l+1}\left(a^E_{lm}\vec{X}_{lm} + a^M_{lm}\,\hat{r}\times\vec{X}_{lm}\right)\right|^2 \\ &= \frac{Z_0}{2k^2}\left|\sum_{lm} (-i)^{l+1}\left(a^E_{lm}\vec{X}_{lm}\times\hat{r} + a^M_{lm}\vec{X}_{lm}\right)\right|^2 \end{aligned}}$$

$$(9.150)$$