2|17|2016 $\vec{H} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a^E_{lm} f_l(kr)\vec{X}_{lm} - \frac{i}{k}\vec{\nabla}\times(g_l\vec{X}_{lm})\right]$

$\cancel{\vec{E} = \frac{Z_0 i}{k}(f(k\hat{r}\times\vec{H})}$ $\vec{E} = -\frac{Z_0}{ik}\vec{\nabla}\times\vec{H}$

$\vec{E} = Z_0\sum_{lm}\left[\frac{i}{k}\vec{\nabla}\times\left(a^E_{lm} f_l\vec{X}_{lm}\right) + a^m_{lm}\; g_l(kr)\vec{X}_{lm}\right]$

$f = g = h_l^{(1)}(kr)$

$kr \gg 1, (l)$ $\quad h \to (-i)^{l+1}\frac{e^{ikr}}{kr}$

$\frac{dP}{d\Omega} = \frac{1}{2}\frac{|\vec{H}|^2}{Z_0} = \frac{1}{2}Z_0|\vec{E}|^2$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{l,m}(-i)^{l+1}\left(a^E_{lm}\vec{X}_{lm} + a^m_{lm}\hat{r}\times\vec{X}_{lm}\right)\right.$$
$$\left.\underbrace{\left(a^E_{lm}\vec{X}_{lm}\times\hat{r} + a^m_{lm}\vec{X}_{lm}\right)}_{\text{polarization }\vec{\epsilon}}\right|^2$$

Pure mode (E or m) $\to \frac{dP_{lm}}{d\Omega} = |a_{lm}|^2\frac{Z_0}{2k^2}|\vec{X}_{lm}|^2$

JDJ. $\vec{L} = L_x \hat{x} + L_y \hat{y} + L_z \hat{z}$

$$= \frac{1}{2}(L_+ + L_-)\hat{x} + \frac{1}{2i}(L_+ - L_-)\hat{y} + L_z \hat{z}$$

$$= \frac{1}{2} L_+ (\hat{x} + i\hat{y}) + \frac{1}{2} L_- (\hat{x} - i\hat{y}) + L_z \hat{z}$$

$$\boxed{|\vec{X}_{lm}|^2 = \frac{\frac{1}{2}|L_+ Y_{lm}|^2 + \frac{1}{2}|L_- Y_{lm}|^2 + m^2 |Y_{lm}|^2}{l(l+1)}}$$

But, polarization direction –

$$\boxed{\vec{L} = \frac{1}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)}$$

$l, m = (1, 0)$ $\quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$

$$\vec{X}_{10} = \frac{1}{\sqrt{2}}\sqrt{\frac{3}{4\pi}}\frac{1}{i}\hat{\phi}(-\sin\theta) = \sqrt{\frac{3}{8\pi}}\, i\hat{\phi}\sin\theta$$

$$\boxed{|\vec{X}_{10}|^2 = \frac{3}{8\pi}\sin^2\theta}$$

$\hat{r} \times \vec{m} = \sin\theta\,\hat{\phi}$ ($\hat{z}$) magnetic dipole

$(\hat{r} \times \vec{p}) \times \hat{r} = \sin\theta\,\hat{\theta}$ electric dipole.

$$\vec{X}_{1,1} = \frac{1}{\sqrt{2}}\left(-\sqrt{\frac{3}{8\pi}}\right)\frac{1}{i}\left(\hat{\phi}\cdot\cos\theta - \frac{\hat{\theta}}{\cancel{\sin\theta}}\cancel{\sin\theta}\,(i)\right) e^{i\phi}$$

$$\boxed{|\vec{X}_{11}|^2 = \sqrt{\frac{3}{16\pi}}\,(1 + \cos^2\theta)}$$

$\vec{p} \cdot \hat{x} + i\hat{y}$

rotating.

$$\vec{X}_{l,0} = \frac{1}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\,\frac{1}{i}\,\hat{\phi}\,(-\sin\theta)\,P_l'(\cos\theta)$$

$$|\vec{X}_{l,0}|^2 = \frac{2l+1}{4\pi l(l+1)}\sin^2\theta\left[P_l'(\cos\theta)\right]^2$$

$$\vec{X}_{l,l} \sim \hat{\phi}\left[l(\sin\theta)^{l-1}\cdot\cos\theta - \frac{\hat{\theta}}{\sin\theta}\,il\,\sin^l\theta\right]e^{il\phi}$$

$$|\vec{X}_{l,l}|^2 \sim \sin^{2(l-1)}\theta\cdot(1+\cos^2\theta)$$

$$Y_{l,1} \sim \sin\theta\, P_l'(\cos\theta)$$

$$\vec{X}_{l,1} \sim \cos\theta\, P_l'(\cos\theta) + (\sin\theta)^2\ldots$$

only $m=1$ is nonzero at $\theta=0$

$$\vec{X}_{2,0} = \frac{1}{\sqrt{6}}\left(\sqrt{\frac{3}{4\pi}}\cdot\frac{1}{i}\,\hat{\phi}\,\frac{\partial}{\partial\theta}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\right)$$

$$= \frac{1}{\sqrt{8\pi}}\,\frac{3}{2i}\left(-\sin\theta\cos\theta\,\hat{\phi}\right)$$

$$|\vec{X}| \longrightarrow \sin^2\theta\cos^2\theta \quad \text{Eq. (?)}$$

$$\int d\Omega \, \sin^2\theta = \frac{8\pi}{3}. \qquad \int d\Omega \, \frac{1}{2}(1+\cos^2\theta) = \frac{8\pi}{3}.$$

$$\int d\Omega \, \vec{X}^*_{lm} \cdot \vec{X}_{l'm'} = \frac{1}{\sqrt{l(l+1)}} \frac{1}{\sqrt{l'(l'+1)}} \int d\Omega \, (\vec{L} Y_{lm})^* \cdot (\vec{L} Y_{l'm'})$$

ϕ: integrate by parts $\rightarrow (-)$, periodic

θ: $\cancel{\left(\frac{1}{i}\right)^* = \frac{-1}{i}}$ & $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) \rightarrow \mathbf{L}^2$

$$\left(\frac{1}{i}\right)^* = -\frac{1}{i}$$

$$\int d\Omega \, \vec{X}^*_{lm} \cdot \vec{X}_{l'm'} = \frac{l(l+1)}{\sqrt{l(l+1)}\sqrt{l'(l'+1)}} \delta_{ll'} \delta_{mm'} = \delta_{ll'} \delta_{mm'}$$

$$\int d\Omega \, (\hat{r} \times \vec{X})^* \cdot (\hat{r} \times \vec{X}')$$

$$\hat{r} \cdot \left(\vec{X}^* \times (\hat{r} \times \vec{X}')\right) = \hat{r} \cdot \left(\hat{r}(\vec{X}^* \cdot \vec{X}') - \vec{X}'(\hat{r} \cdot \vec{X}^*)\right)$$

$$= \vec{X}^*_{lm} \cdot \vec{X}_{l'm'} \quad \text{same}$$

$$\int d\Omega \, \vec{X}^*_{lm} \cdot (\hat{r} \times \vec{X}_{l'm'}) = 0$$

$$\vec{L} \cdot (\hat{r} \times \vec{L}) = (\vec{L} \times \hat{r}) \cdot \vec{L} = 0.$$

$$P = \int d\Omega \, \frac{Z_0}{2k^2} \sum_{l,m} \sum_{l'm'} (+i)^{l+1} (-i)^{l'+1} \left(a^E_{lm} \vec{X}_{lm} + a^M_{lm} \hat{r} \times \vec{X}_{lm} \right)^* \cdot \left(a^E_{l'm'} \vec{X}_{l'm'} + \cdots \right)$$

squares $\to \delta_{ll'} \delta_{mm'}$ cross $\to 0$

$$\boxed{P = \frac{Z_0}{2k^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(|a^E_{lm}|^2 + |a^M_{lm}|^2 \right)}$$

(9.155)

$$\langle u \rangle = \frac{1}{4} \varepsilon_0 |E|^2 + \frac{1}{4} \mu_0 |H|^2 = \frac{1}{2} \mu_0 |H|^2$$

$$U = \int r^2 dr \, d\Omega \, u$$

$$\frac{dU}{dr} = \int r^2 d\Omega \, \frac{1}{2} \mu_0 |H|^2$$

$$\boxed{\frac{dU}{dr} = \frac{1}{2} \frac{\mu_0}{k^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(|a^E_{lm}|^2 + |a^M_{lm}|^2 \right)}$$

$$\frac{P}{U'} = \frac{(dE/dt)}{(dE/dr)} = v = \frac{Z_0}{\mu_0} = \frac{1}{\mu_0} \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \underline{\underline{c}}$$

Angular momentum

$$\vec{m} = \frac{1}{2c^2}\vec{r}\times(\vec{E}\times\vec{H}^*) \qquad \vec{\nabla}\times\vec{H} = \frac{-ik}{Z_0}\vec{E}$$

$$\vec{m} = \frac{1}{2c^2}\vec{r}\times\left[\left(\frac{-Z_0}{ik}\right)(\vec{\nabla}\times\vec{H})\times\vec{H}^*\right]$$

$$= \left(\frac{-Z_0}{ik}\right)\left(\frac{1}{2c^2}\right)\left[(\vec{\nabla}\times\vec{H})\cancel{(\vec{r}\cdot\vec{H}^*)} - \vec{H}^*\left(\vec{r}\cdot(\vec{\nabla}\times\vec{H})\right)\right]$$

$$= \frac{Z_0}{2kc^2}\vec{H}^*\left[\frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{H}\right] = \frac{\mu_0}{2\omega}\vec{H}^*(\vec{L}\cdot\vec{H})$$

$$\frac{d\vec{M}}{dr} = \int r^2 d\Omega \frac{1}{k^2r^2}\frac{\mu_0}{2\omega}\sum_{lm}\sum_{l'm'}(+i)^{l+1}(-i)^{l'+1}$$
$$\times\left(a^E_{lm}\vec{X}_{lm} + a^M_{lm}\hat{r}\times\vec{X}_{lm}\right)^*\left(a^{\prime}_{l'm'}\vec{L}\cdot\cancel{\vec{X}}_{l'm'} + \cdots\right)$$

Electric mode

$$= \frac{\mu_0}{2\omega k^2}\int d\Omega \sum_{ll'mm'}(+i)^{l+1}(-i)^{l'+1}$$
$$\left(a^E_{lm}\frac{1}{\sqrt{l(l+1)}}\vec{L}Y_{lm}\right)^*\cdot\left(a^E_{l'm'}\,l'(l'+1)Y_{l'm'}\right)$$

$l'=l$ $\quad$ $m' = m\pm1$ $\quad$ [L_z diagonal $\to m$

$$\frac{dM_z}{dr} = \frac{\mu_0}{2\omega k^2} \sum_{l,m} m\left(|a^E_{lm}|^2 + |a^M_{lm}|^2 \right)$$

single mode (single m)

$$\frac{dM_z/dr}{dU/dr} = \frac{M_z}{U} = \frac{m \cdot \frac{\mu_0}{2\omega k^2} |a|^2}{\frac{\mu_0}{2k^2} |a|^2}$$

$$= \frac{m}{\omega} = \frac{m\hbar}{\hbar\omega}$$