$a^{E}(l,m)$

2/19/2010 Multipole expansion $\sum a^{E}_{lm}\, h_l^{(1)}(kr)\, \vec{X}_{lm}$

Last project. relate to sources.

$\vec{\nabla}\cdot\vec{H} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ ← ??

$\vec{\nabla}\times\vec{H} = \vec{J} + \underline{\vec{\nabla}\times\vec{M}} + \frac{\partial\vec{D}}{\partial t} = \vec{J} + (\ ) - i\omega\varepsilon_0\vec{E}$

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = +i\omega\vec{B} = i\omega\mu_0\vec{H} = \underline{ikZ_0\vec{H}}$.

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = +i\omega\rho$ $\qquad$ [conductor $\vec{E}' = \vec{E}\left(1+\frac{i\sigma}{\varepsilon_0\omega}\right) = \frac{\varepsilon}{\varepsilon_0}\vec{E}$]

Let $\vec{E}' = \vec{E} + \frac{i}{\omega\varepsilon_0}\vec{J}$ $\qquad$ $[\vec{H}' = \vec{H}]$.

$\vec{\nabla}\cdot\vec{E}' = \vec{\nabla}\cdot\vec{E} + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\cdot\vec{J} = \frac{\rho}{\varepsilon_0} + \frac{i}{\omega\varepsilon_0}(i\omega\rho) = \underline{0}$.

$\vec{\nabla}\cdot\vec{E}' = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{H}' = 0$

$\vec{\nabla}\times\vec{E}' = \vec{\nabla}\times\vec{E} + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J} = ikZ_0\vec{H}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}$

$\vec{\nabla}\times\vec{H}' = \vec{J} - i\omega\varepsilon_0\vec{E} = -i\omega\varepsilon_0\left(\vec{E} + \frac{i}{\omega\varepsilon_0}\vec{J}\right) = -\frac{ik}{Z_0}\vec{E}'$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}') = \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{H}'}) - \nabla^2\vec{H}'$$

$$= \vec{\nabla}\times\left(-\frac{ik}{Z_0}\vec{E}'\right) = -\frac{ik}{Z_0}\left(ikZ_0\vec{H}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}\right)$$

$$\boxed{(\nabla^2+k^2)\vec{H}' = -\vec{\nabla}\times\vec{J}}$$

$$(\nabla^2+k^2)\vec{A} = -\mu_0\vec{J} =$$

$$(\nabla^2+k^2)\vec{B} = -\mu_0(\vec{\nabla}\times\vec{J})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}') = \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}'}) - \nabla^2\vec{E}'$$

$$= \vec{\nabla}\times\left(ikZ_0\vec{H}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}\right) = ikZ_0\left(-\frac{ik}{Z_0}\right)\vec{E}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{\nabla}$$

$$\boxed{(\nabla^2+k^2)\vec{E}' = -\frac{iZ_0}{k}\vec{\nabla}\times(\vec{\nabla}\times\vec{J})}$$

Recall: $\quad \nabla^2(\vec{r}\cdot\vec{v}) = 2(\vec{\nabla}\cdot\vec{v}) + \vec{r}\cdot(\nabla^2\vec{v})$

$$\boxed{(\nabla^2+k^2)(\vec{r}\cdot\vec{H}') = -\vec{r}\cdot(\vec{\nabla}\times\vec{J}) = -i\vec{L}\cdot\vec{J}}$$

$$\boxed{(\nabla^2+k^2)(\vec{r}\cdot\vec{E}') = +\frac{Z_0}{k}\vec{L}\cdot(\vec{\nabla}\times\vec{J})}$$

$$(\hat{r}\cdot\vec{E}') = -\int d^3x' \frac{1}{4\pi} \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \frac{Z_0}{k} \vec{L}'\cdot(\vec{\nabla}'\times\vec{J})$$

$$= -\int d^3x' \sum_{l=0}^{\infty}\sum_{m=-l}^{l} ik\, j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y_{lm}^{*}(\theta',\phi')\, \underline{Y_{lm}(\theta,\phi)}$$

$r_< \to r'$, $r_> \to r$

$$\times \frac{Z_0}{k} \vec{L}'\cdot(\vec{\nabla}'\times\vec{J})$$

(9.123)

$$-\frac{k}{\sqrt{l(l+1)}}\int d\Omega\, Y_{lm}^{*}(\theta,\phi)\,(\hat{r}\cdot\vec{E}) = Z_0\, a_{lm}^{E}\, h_l^{(1)}(kr)$$

$$\boxed{a_{lm}^{E} = \frac{ik}{\sqrt{l(l+1)}}\int d^3x'\, j_l(kr')\, Y_{lm}^{*}(\theta',\phi')\, \vec{L}'\cdot(\vec{\nabla}'\times\vec{J})}$$

(9.165)

(9.123)

$$(\hat{r}\cdot\vec{H}') = +\int d^3x' \sum_{lm} ik\, j_l(kr')\, h_l^{(1)}(kr)\, Y_{lm}^{*}(\theta',\phi')\, \underline{Y_{lm}(\theta,\phi)}$$

$$\times\ i\vec{L}'\cdot\vec{J}$$

$$\frac{k}{\sqrt{l(l+1)}}\int d\Omega\, Y_{lm}^{*}(\theta,\phi)\,(\hat{r}\cdot\vec{H}) = a_{lm}^{M}\, h_l^{(1)}(kr)$$

$$\boxed{a_{lm}^{M} = \frac{-k^2}{\sqrt{l(l+1)}}\int d^3x'\, j_l(kr')\, Y_{lm}^{*}(\theta',\phi')\, \vec{L}'\cdot\vec{J}}$$

(9.165)

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Familiar (?) forms $\rightarrow$ under $\vec{L}$

$$\vec{L}\cdot\vec{J} = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{J} = \frac{1}{i}\vec{r}\cdot(\vec{\nabla}\times\vec{J}) = \frac{1}{i}\underline{\vec{\nabla}\cdot(\vec{r}\times\vec{J})}$$

$$\vec{L}\cdot(\vec{\nabla}\times\vec{J}) = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot(\vec{\nabla}\times\vec{J})$$

$$= \frac{1}{i}\vec{r}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{J})) = \frac{1}{i}\vec{r}\cdot\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \nabla^2\vec{J}\right]$$

$$= \frac{1}{i}\left[r\frac{\partial}{\partial r}(\vec{\nabla}\cdot\vec{J}) - \vec{r}\cdot(\nabla^2\vec{J})\right]$$

$$= \frac{1}{i}\left(r\frac{\partial}{\partial r}(\vec{\nabla}\cdot\vec{J})\right) - \frac{1}{i}\left(\nabla^2(\vec{r}\cdot\vec{J}) - 2(\vec{\nabla}\cdot\vec{J})\right)$$

$$= i\nabla^2(\vec{r}\cdot\vec{J}) - ir\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)(\vec{\nabla}\cdot\vec{J})$$

$\hookrightarrow i\omega\rho = ick\rho$

$$a^E_{\ell m} = \frac{ik^2}{\sqrt{\ell(\ell+1)}}\int d^3x'\, j_\ell(kr')Y^*_{\ell m}(\theta',\phi') \times\left[\frac{r'}{r'^2}\frac{\partial}{\partial r'}(r'^2 c\rho) - \nabla'^2(\vec{r}'\cdot\vec{J})\right]$$

$\hookleftarrow$ parts $\qquad \hookleftarrow$ parts (k^2)

$$a^E_{\ell m} = \frac{-ik^2}{\sqrt{\ell(\ell+1)}}\int d^3x'\, Y^*_{\ell m}(\theta',\phi')\left[c\rho\frac{\partial}{\partial r'}(r' j_\ell(kr')) - ik\, j_\ell(kr')(\vec{r}'\cdot\vec{J})\right]$$

(9.167)

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$$a^{M}_{\ell m} = \frac{-ik^2}{\sqrt{\ell(\ell+1)}} \int d^3x' \, Y^*_{\ell m}(\theta', \phi') \, j_\ell(kr') \, \vec{\nabla}' \cdot (\vec{r}' \times \vec{J}) \qquad (9.168)$$

exact ↑

Long wavelength. $kd \ll 1$. $j_\ell \to \frac{(kr')^\ell}{(2\ell+1)!!}$ $\quad k(\vec{r}\cdot\vec{J}) \ll c\rho$.

$$a^{E}_{\ell m} \to \frac{-ik^2}{\sqrt{\ell(\ell+1)}} \int d^3x' \, Y^*_{\ell m}(\theta', \phi') \, c\rho(\vec{x}') \, \frac{(\ell+1)(kr')^\ell}{(2\ell+1)!!}$$

$$a^{E}_{\ell m} = \frac{-ick^{\ell+2}}{(2\ell+1)!!} \sqrt{\frac{\ell+1}{\ell}} \int d^3x' \, Y^*_{\ell m}(\theta', \phi') \, r'^\ell \rho$$

$q_{\ell m} = Q_{\ell m}$

$$a^{M}_{\ell m} = \frac{ik^{\ell+2}}{(2\ell+1)!!} \sqrt{\frac{\ell+1}{\ell}} \left(\frac{-1}{\ell+1} \int d^3x' \, Y^*_{\ell m}(\theta', \phi') \, r'^\ell \, \vec{\nabla}' \cdot (\vec{r}' \times \vec{J}) \right)$$

$M_{\ell m}$.

+ magnetization