recall: $\vec{\nabla}\times\left(f(r)\vec{X}\right) = \frac{1}{r}\frac{\partial}{\partial r}(rf)\,\hat{r}\times\vec{X} + i\sqrt{\ell(\ell+1)}\,\frac{f}{r}Y_{\ell m}\,\hat{r}$

α-term

$$\frac{1}{2}\alpha\, h^{(1)} + \left(\frac{1}{2}h^{(1)} + \frac{1}{2}h^{(2)}\right)$$

$$= i\frac{Z_s}{Z_0}\frac{1}{x}\frac{\partial}{\partial x}\left(x\left(\frac{1}{2}\alpha_\ell h^{(1)} + \frac{1}{2}h^{(1)} + \frac{1}{2}h^{(2)}\right)\right)$$

$cB = Z_0 H$

$$(1+\alpha_\ell)\left[h_\ell^{(1)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}\left(x h^{(1)}\right)\right]$$

$$= -h^{(2)} + i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}\left(x h^{(2)}\right)$$

$$\boxed{1+\alpha_\ell = -\left.\frac{h_\ell^{(2)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}\left(x h^{(2)}\right)}{h_\ell^{(1)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}\left(x h^{(1)}\right)}\right|_{x=ka}}$$

for e: duality. $\vec{E}\to c\vec{B}\to Z_0\vec{H}$

$c\vec{B}\to -\vec{E}$

$$\frac{Z_s}{Z_0}\to\frac{Z_0}{Z_s}$$

$$\hat{r}\times\vec{E}\big|_s = Z_s\left(\hat{r}\times(\hat{r}\times\vec{H})\right)$$

$$= Z_s\left[\hat{r}\left(\hat{r}\cdot\vec{H}_\parallel\right) - \vec{H}_\parallel(\hat{r}\cdot\hat{r})\right]$$

$$\left.\vec{H}_\parallel = -\frac{1}{Z_s}\hat{r}\times\vec{E}_\parallel\right|_s$$

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$$\boxed{1+\alpha_\ell = - \left. \frac{h_\ell^{(2)} - i \frac{Z_s}{Z_0} \frac{1}{x} \frac{d}{dx}\left(x h^{(2)}\right)}{h_\ell^{(1)} - i \frac{Z_s}{Z_0} \frac{1}{x} \frac{d}{dx}\left(x h^{(1)}\right)} \right|_{x = ka}}$$

β_ℓ: $(Z_s/Z_0) \longrightarrow (Z_0/Z_s)$. (10.66)

$$h^{(2)} = j - i n = h^{(1)*}$$

$Z_s = 0$

$Z_s \to \infty$

Z_s pure imaginary

phase shifts

$$\boxed{1+\alpha_\ell = \frac{(R e^{i\delta})}{(R e^{-i\delta})} = e^{2i\delta_\ell}}$$

Conducting. $Z_s = 0$

$$1+\alpha_\ell = -\frac{h^{(2)}}{h^{(1)}} = -\frac{(j-in)}{(j+in)} = -\frac{(j^2-n^2)-2ijn}{j^2+n^2}$$

$$= \frac{n^2-j^2+2ijn}{n^2+j^2} = e^{2i\delta_\ell} = \cos 2\delta_\ell + i \sin 2\delta_\ell$$

$$\cos\delta_\ell = \frac{n}{\sqrt{n^2+j^2}} \qquad \sin\delta_\ell = \frac{j}{\sqrt{n^2+j^2}} \qquad \tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}$$

$$1+\beta_\ell = e^{2i\delta_\ell'} \qquad \tan\delta_\ell' = \left. \frac{\frac{1}{x}\frac{d}{dx}\left(x j_\ell(x)\right)}{\frac{1}{x}\frac{d}{dx}\left(x n_\ell(x)\right)} \right|_{x=ka} \qquad (10.68)$$

$ka \ll 1$. $\quad \tan\delta_\ell = -\dfrac{(ka)^{2\ell+1}}{(2\ell-1)!!\,(2\ell+1)!!}$

$$\tan\delta_\ell' = \frac{\ell+1}{\ell}\,\frac{(ka)^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!}$$

$\alpha_\ell = e^{2i\delta_\ell} - 1 \simeq 2i\delta_\ell \qquad \beta_\ell \simeq 2i\,\delta_\ell'$

$\ell = 1 \qquad \delta_1 = -\dfrac{x^3}{3} \qquad \delta_\ell' = \dfrac{2}{3}x^3$

$\boxed{\alpha_1 = -\dfrac{2i}{3}(ka)^3} \qquad \boxed{\beta_1 = +\dfrac{4i}{3}(ka)^3}$

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{2k^2}\cdot(3)\left|\left(-\frac{2i}{3}(ka)^3\vec{X}_1\right) \pm i\left(\frac{4}{3}(ka)^3\,\hat{r}\times\vec{X}\right)\right|^2$$

$$= \frac{\pi}{2k^2}\cdot 3\cdot\frac{4}{9}(ka)^6\left|\vec{X}_1 \mp 2i\,\hat{r}\times\vec{X}\right|^2$$

$$\vec{X}_{1,1} = \sqrt{\frac{3}{16\pi}}\left(\hat{\theta} + i\cos\theta\,\hat{\phi}\right)e^{i\phi}$$

$$\hat{r}\times\vec{X} = \sqrt{\frac{3}{16\pi}}\left(-i\cos\theta\,\hat{\theta} + \hat{\phi}\right)e^{i\phi}$$

$(1+2i\cos\theta)(1-2i\cos\theta) = 1+4\cos^2\theta$

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{2k^2}\cdot\frac{4}{3}(ka)^6\cdot\frac{3}{16\pi}\left|\hat{\theta}(1-2\cos\theta) + \hat{\phi}(i\cos\theta - 2i)\right|^2$$

$$= k^4a^6\left(\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right)$$

$\cancel{1+4\cos^2\theta}\quad (1-2\cos\theta)^2 + (\cos\theta - 2)^2$

$5(1+\cos^2\theta) - 8\cos\theta$

3

$$\alpha_\ell = e^{2i\delta_\ell} - 1 = \cos 2\delta_\ell - 1 + i \sin 2\delta_\ell$$

$$= i(2\sin\delta_\ell \cos\delta_\ell) - 2\sin^2\delta_\ell$$

$$= 2i \sin\delta_\ell (\cos\delta_\ell + i\sin\delta_\ell) = \underline{2i\,\delta_\ell\, e^{i\delta_\ell}}$$

$$\sigma_{scatt} = \frac{\pi}{2k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(|\alpha_\ell|^2 + |\beta_\ell|^2\right)$$

$$\boxed{\sigma_{scatt} = \frac{2\pi}{k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(\sin^2\delta_\ell + \sin^2\delta_\ell'\right)}$$

Not in JDJ but should be.

$\ell \gg ka$ $\quad j_\ell \sim \frac{x^\ell}{(2\ell+1)!!} \quad n_\ell \sim -\frac{(2\ell-1)!!}{x^{\ell+1}}$ | dipole.

$$\tan\delta_\ell \sim \frac{x^{2\ell+1}}{(\;)!!(\;)!!} \ll 1.$$

$\ell \lesssim ka$. $\quad ka \sim$ random phase, $\langle \sin^2\delta_\ell \rangle = \frac{1}{2}$.

$\underline{ka \ll 1}$. $\quad \sigma_{sc} \simeq \frac{10\pi}{3} k^4 a^6$

$ka \gg 1$. $\quad \sigma_{sc} \simeq \frac{2\pi}{k^2}\sum_{\ell=1}^{ka}(2\ell+1)\left(\frac{1}{2}+\frac{1}{2}\right)$

$\hookrightarrow 1,\ 1+3=4,\ 1+3+5=9 \Rightarrow (ka)^2$

$$\boxed{\sigma \simeq 2\pi a^2}$$