3/14/16 **Diffraction** §10.5 short wavelength $kd \gg 1$.

start with scalar. $(\nabla^2+k^2)\psi = 0$ $(\psi = \Phi, \vec{A}, \vec{E}, \vec{B})$

Green $$\int_V d^3x \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] = \oint_S d^2a \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)$$

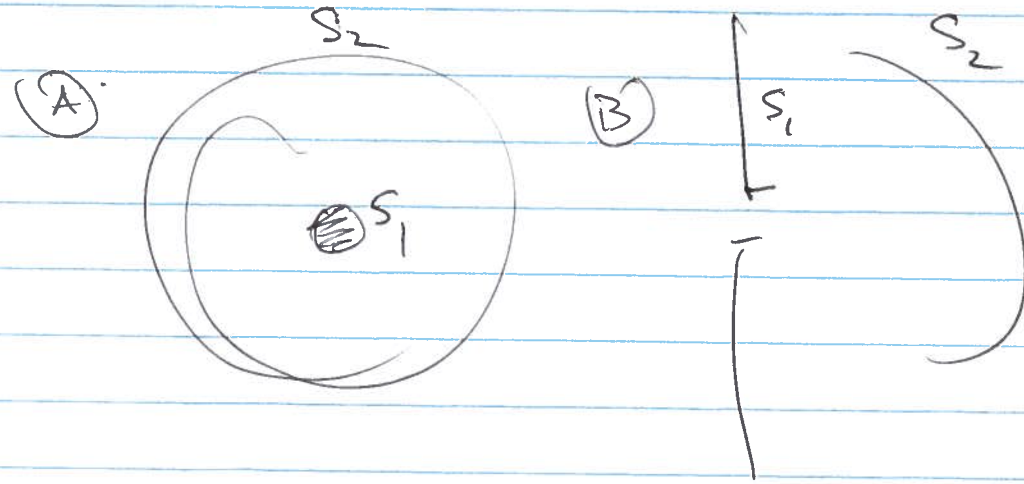
$$(\nabla^2+k^2)G = -\delta(\vec{x}-\vec{x}') \qquad G = \frac{1}{4\pi}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

$\phi = G$

$$\int_V d^3x' \left[G(\cancel{-k^2\psi}) - \psi(\cancel{-k^2G} - \delta(\vec{x}-\vec{x}'))\right] = \oint d^2a'\left(G\frac{\partial\psi}{\partial n'} - \psi\frac{\partial G}{\partial n'}\right)$$

$$\boxed{\psi(\vec{x}) = \oint_S d^2a' \left(G\frac{\partial\psi}{\partial n'} - \psi\frac{\partial G}{\partial n'}\right)}$$

Two kinds of problems



$$S = S_1\,[\text{sources}] \oplus S_2 = \{r \to \infty\}.$$

(on S_2) $\quad \psi \to \frac{e^{ikr}}{r} f(\theta,\phi) \qquad G \to \frac{1}{4\pi}\frac{e^{ikr}}{r}\, e^{-ik\hat{r}\cdot\vec{x}'}$

$$\frac{\partial\psi}{\partial n'} = \hat{r}'\cdot\vec{\nabla}'\psi = ik\psi + O\left(\frac{\psi}{r}\right)$$

$$\frac{\partial G}{\partial n'} = \hat{r}'\cdot\vec{\nabla}' G = ikG + O\left(\frac{G}{r}\right)$$

$$\int_{S_2} d^2a' \left(G\frac{\partial\psi}{\partial n'} - \psi\frac{\partial G}{\partial n'}\right) = \int_{r\to\infty} r^2 d\Omega \left[\frac{1}{4\pi}\frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'} \cdot ik\frac{e^{ikr}}{r} f - \frac{ik}{4\pi}\frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'}\frac{e^{ikr}}{r} f + O\left(\frac{1}{r}\right)\right] \to 0.$$

(*) Reverse $\hat{n}'$ (into V $\Leftrightarrow$ out of screen)

(*) $\vec{R} = \vec{x} - \vec{x}' \qquad \vec{\nabla}' R = -\vec{\nabla} R = -\hat{R}$

$$\boxed{\psi(\vec{x}) = -\frac{1}{4\pi}\int_{S_1} d^2a' \, \frac{e^{ikR}}{R}\left(\hat{n}'\cdot\vec{\nabla}'\psi + \underbrace{ik}_{\left(1+\frac{i}{kR}\right)}\hat{n}'\cdot\hat{R}\,\psi\right)}$$

Kirchhoff integral formula (10.79).

useful formula, $\cancel{\frac{1}{kR}} \to 0$. ($r \to$ large).

$$\boxed{\psi_0 = e^{i\vec{k}_0 \cdot \vec{x}}}$$

$$\boxed{G = \frac{1}{4\pi}\frac{e^{ikR}}{R} \to \frac{1}{4\pi}\frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'}}$$

$$\boxed{S_1 = \{\text{screen with openings}\}}$$

Approximate: $\psi = 0$ "on screen"
$\psi = \psi_0$ "on openings"

$$\psi = -\frac{1}{4\pi}\int_{\substack{S_i \\ \text{openings}}} d^2a'\, \hat{n}' \cdot \left(i\vec{k}_0 + ik\hat{r} \right) e^{i\vec{k}_0\cdot\vec{x}'} \frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'}$$

$\vec{k}_{in} = \vec{k}_0$ $\quad \vec{k}_{out} = k\hat{r}$.

$$\boxed{\psi = -\frac{ik}{4\pi}\int_{\text{openings}} d^2a' \left(\cos\theta_{in} + \cos\theta_{out} \right) e^{i(\vec{k}_{in} - \vec{k}_{out})\cdot\vec{x}'} \frac{e^{ikr}}{r}}$$

~~10.86~~

Objection! Both ψ, $\frac{\partial\psi}{\partial n}$ specified $\to$ over specified.

Fix! $G_N, G_D = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} \pm \frac{e^{ikR'}}{R'}\right)$ image.

$G_D|_S = 0$ $\frac{\partial G_N}{\partial S}\Big|_S = 0$ $\to$ other term doubles.

$(\cos\theta_{in} + \cos\theta_{out}) \to \boxed{2\cos\theta_{in}}$ or $\boxed{2\cos\theta_{out}}$

$\theta_{in} = \theta_{out} + O\left(\frac{1}{ka}\right)$.

Why is this a reasonable thing to do?

Look at $\boxed{\dfrac{dy}{dx} = y\,, \quad y(0) = 1}$

$$\int_0^x \left(\frac{dy}{dx'}\right) dx' = y(x) - y(0) = \int_0^{x} y(x')\,dx'$$

$$\boxed{y(x) = 1 + \int_0^x y(x')\,dx'}$$

Diff. eq. + B.C.
$\rightarrow$ integral equation

Iterate

$$\boxed{y_0 = 0}$$

$$y_1 = 1 + \int_0^x y_0\,dx = 1$$

$$y_2 = 1 + \int_0^x y_1\,dx' = 1 + x$$

$$y_3 = 1 + \int_0^x y_2\,dx' = 1 + x + \frac{x^2}{2}$$

$$y_4 = 1 + \int_0^x y_3\,dx' = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$\longrightarrow \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x} \checkmark$$

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"contraction map" $\quad$ $x \to fx$. $\quad d(fx, fy) \leq \lambda\, d(x,y)$ $\quad (\lambda < 1)$

converges to unique fixed point

$$d(f^n x, f^n y) \leq \lambda^n d(x,y) \to 0 .$$

Ex. opening: $\left(-\frac{a}{2} < x < \frac{a}{2}\right)$, $\left(-\frac{b}{2} < y < \frac{b}{2}\right)$ $\quad b \to \infty$

$$\psi = -\frac{ik}{4\pi} \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy' \; \cancel{e^{ik(\ldots)\cos\theta}}\, (2)\, e^{-ik_x x'} e^{-ik_y y'}$$

$\int dy' \to \delta(k_y) \to k\sin\theta\ \cancel{\sin\phi} \to \phi = 0 \to \int = b$

$$\psi = -\frac{b}{2\pi} \int_{-a/2}^{a/2} dx'\, e^{-ik_x x'} = \left.\frac{e^{-ik\sin\theta x'}}{-ik\sin\theta}\right|_{-a/2}^{a/2}$$

$$= \frac{e^{i\frac{ka}{2}\sin\theta} - e^{-i\frac{ka}{2}\sin\theta}}{ik\sin\theta} = \frac{a}{\frac{1}{2}ka\sin\theta}\left(\frac{e^{i\frac{ka}{2}\sin\theta} - e^{-i\frac{ka}{2}\sin\theta}}{2i}\right)$$

$$\boxed{\psi = \frac{-iabk}{2\pi} \frac{\sin\left(\frac{1}{2}ka\sin\theta\right)}{\left(\frac{1}{2}ka\sin\theta\right)}}$$

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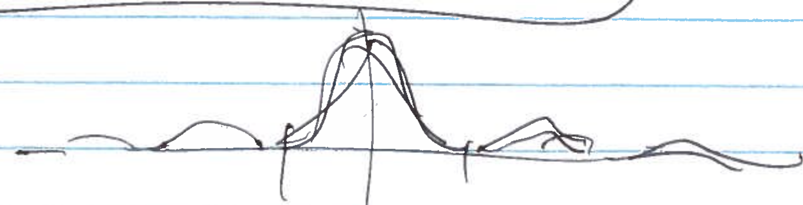
$\vec{S} = ?$ $\quad I \propto |\psi|^2$

$E \propto \omega^2 |\psi|^2$

$\vec{S} \propto \omega \vec{k} |\psi|^2 c_0^2)$.

$$\boxed{I = I_0 \cdot \frac{\sin^2\left(\frac{1}{2} ka \sin\theta\right)}{\left(\frac{1}{2} ka \sin\theta\right)^2}}$$



$\cancel{\theta = \pi}$

$\frac{1}{2} ka \sin\theta = \pi n$

$\langle\theta^2\rangle =$

$\sin\theta = \frac{2\pi n}{ka} = \frac{2\pi}{a}\left(\frac{\lambda}{2\pi}\right) n$

$n\lambda = a \cdot \sin\theta$

$$\int_{-n\pi}^{n\pi} dx \, \frac{\sin^2 x}{x^2}$$

$(-\infty, \infty) \rightarrow \pi$

$(-\pi, \pi) \rightarrow 2\,\mathrm{Si}(2\pi) \rightarrow 0.902823$

$(-2\pi, 2\pi) \rightarrow 2\,\mathrm{Si}(4\pi) \rightarrow 0.949939$

$(-3\pi, 3\pi) \rightarrow 2\,\mathrm{Si}(6\pi) \rightarrow 0.966410$

0.974748

$1 - \frac{1}{\pi^2 n}$