3/18/2016. Vector Green's Theorem (Hirose) (JDJ § 10.6)

goal = JDJ (10.89).

$$\vec\nabla\cdot\left[\vec E\times(\vec\nabla\times\vec F)\right] = (\vec\nabla\times\vec E)\cdot(\vec\nabla\times\vec F) - \vec E\cdot\left(\vec\nabla\times(\vec\nabla\times\vec F)\right)$$

$$\vec\nabla\cdot(\vec a\times\vec b) = (\vec\nabla\times\vec a)\cdot\vec b - \vec a\cdot(\vec\nabla\times\vec b)$$

$$\int_V d^3x'\,\vec\nabla'\cdot\left(\vec E\times(\vec\nabla\times\vec F)\right) = \oint_S da'\,\hat n'\cdot\left[\vec E\times(\vec\nabla'\times\vec F)\right]$$

$$= \int_V d^3x'\left[(\vec\nabla'\times\vec E)\cdot(\vec\nabla'\times\vec F) - \vec E\cdot\left(\vec\nabla'\times(\vec\nabla'\times\vec F)\right)\right]$$

Exchange $\vec E\leftrightarrow\vec F$, subtract

$$\int_V d^3x'\left[\vec E\cdot\left(\vec\nabla'\times(\vec\nabla'\times\vec F)\right) - \vec F\cdot\left(\vec\nabla'\times(\vec\nabla'\times\vec E)\right)\right]$$

$$= \oint_S da'\,\hat n'\cdot\left[\vec F\times(\vec\nabla'\times\vec E) - \vec E\times(\vec\nabla'\times\vec F)\right]$$

$$\vec\nabla\times(\vec\nabla\times\vec F) = \vec\nabla(\vec\nabla\cdot\vec F) - \nabla^2\vec F$$

$$\vec E\cdot\left(\vec\nabla\times(\vec\nabla\times\vec F)\right) - \vec F\cdot\left(\vec\nabla\times(\vec\nabla\times\vec E)\right) =$$

$$= \vec F\cdot(\nabla^2\vec E) - \vec E\cdot(\nabla^2\vec F) + (\vec E\cdot\vec\nabla)(\vec\nabla\cdot\vec F) - (\vec F\cdot\vec\nabla)(\vec\nabla\cdot\vec E)$$

$$= \vec\nabla\cdot\left[\vec E(\vec\nabla\cdot\vec F) - \vec F(\vec\nabla\cdot\vec E)\right] \Rightarrow \text{surface}.$$

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$$\int_V d^3x' \left[\vec{F}\cdot(\nabla^2\vec{E}) - \vec{E}\cdot(\nabla^2\vec{F})\right]$$

$$= \oint_S d^2a' \Big\{ \hat{n}'\cdot\left[\vec{F}(\nabla'\cdot\vec{E}) - \vec{E}(\nabla'\cdot\vec{F})\right]$$

$$- (\hat{n}'\times(\nabla'\times\vec{E}))\cdot\vec{F} - (\hat{n}'\times\vec{E})\cdot(\nabla'\times\vec{F}) \Big\}$$

$$\hat{n}'\cdot\left(\vec{F}\times(\nabla'\times\vec{E})\right) = -\hat{n}'\cdot\left((\nabla'\times\vec{E})\times\vec{F}\right) = -\left(\hat{n}'\times(\nabla'\times\vec{E})\right)\cdot\vec{F}$$

$$\hat{n}'\cdot\left(\vec{E}\times(\nabla'\times\vec{F})\right) = (\hat{n}'\times\vec{E})\cdot(\nabla'\times\vec{F})$$ (symmetry broken)

Now: $\vec{E}$ = Electric field $\quad (\nabla^2+k^2)\vec{E}=0 \quad \nabla\cdot\vec{E}=0$

$\nabla\times\vec{E} = i\omega\vec{B}$

$$\vec{F} = \vec{a}\,G \qquad G = \frac{1}{4\pi}\frac{e^{ikR}}{R} \qquad (\nabla^2+k^2)G = -\delta(\vec{R})$$

$$\text{L.H.S.} \rightarrow \int_V d^3x' \left[\vec{a}G\cdot(\nabla^2\vec{E}) - (\vec{E}\cdot\vec{a})(\nabla^2 G)\right]$$

$$= \vec{a}\cdot\int_V d^3x' \left[G(-k^2\vec{E}) - \vec{E}(-k^2 G - \delta(\vec{x}-\vec{x}'))\right]$$

$$= \vec{a}\cdot\vec{E}(\vec{x})$$

RHS → $$\oint_S d^2a' \Big[\cancel{\ \ }^{\,0} - (\hat{n}'\cdot\vec{E})\,(\vec{a}\cdot\vec{\nabla}'G) - \hat{n}'\times(i\omega\vec{B})\cdot(\vec{a}G) - (\hat{n}'\times\vec{E})\cdot(\vec{a}\times\vec{\nabla}'G)\Big]$$

$$= -\vec{a}\cdot\oint_S d^2a' \Big[(\hat{n}'\cdot\vec{E})\vec{\nabla}'G + (\hat{n}'\times i\omega\vec{B})G + (\hat{n}'\times\vec{E})\times\vec{\nabla}'G\Big]$$

Holds for all $\vec{a}$ →

$$\boxed{\vec{E}(\vec{x}) = -\oint_S d^2a' \Big[(\hat{n}'\cdot\vec{E})\vec{\nabla}'G + \hat{n}\times(i\omega\vec{B})\,G + (\hat{n}'\times\vec{E})\times\vec{\nabla}'G\Big]}$$

$\hat{n}_{\text{outb}} \to -\hat{n}_{\text{into V}}$ → (10.89)

- Diffraction from object (conducting sphere)

- $S = S_1 \oplus S_2 = \{r=a\} \oplus \{r\to\infty\}$

- $\vec{E} = \vec{E}_0 + \vec{E}_{sc}$, $\vec{B} = \vec{B}_0 + \vec{B}_{sc}$

- $\vec{E}_0, \vec{B}_0$ in full integral → $\vec{E}_0, \vec{B}_0$

$\vec{E}_{sc} \to \frac{e^{ikr}}{r}\vec{F}(\theta,\phi)$. $\int_{S_2} d^2a' \,[\vec{E}_{sc}] \to 0$.

Radiation regime $\vec{\nabla}' G \to -ik\hat{r}\, G$

$\vec{R} = \vec{x} - \vec{x}'$

$$\vec{E}_{sc} = \frac{1}{4\pi}\int_{S_1} d^2a' \, \frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{x}'} \left[-(\hat{n}'\cdot\vec{E}_{sc})\, ik\hat{r} + \hat{n}' \times (i\omega\vec{B}_{sc}) - (\hat{n}'\times\vec{E}_{sc})\times ik\hat{r} \right]$$

We know (hope) $\vec{E}_{sc}\cdot\hat{r} = 0$

So must (partly) cancel w/ $\vec{B}_{sc}$ term.

so: project $\vec{\varepsilon}^{*}\cdot$. $\vec{\varepsilon}^{*}\cdot\hat{r} = 0$.

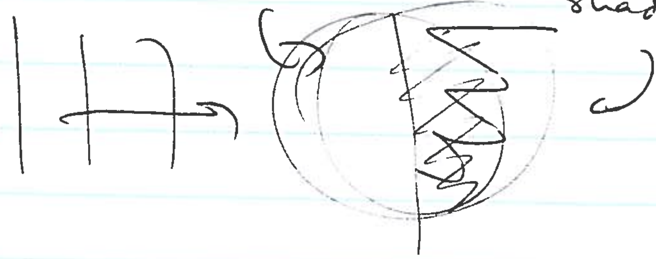
$$\vec{E}_{sc} = \frac{e^{ikr}}{r}\vec{F}_{sc}.$$

$$\vec{\varepsilon}^{*}\cdot\vec{F}_{scatt} = \frac{ik}{4\pi}\int_{S_1} d^2a' \, \vec{\varepsilon}^{*}\cdot\left[(\hat{n}'\times c\vec{B}_{sc}) + \hat{r}\times(\hat{n}'\times\vec{E}_{sc}) \right] \times e^{-ik\hat{r}\cdot\vec{x}'}$$

(10.93).

depends on $\vec{B}_{\parallel}, \vec{E}_{\parallel}\Big|_S$

"effective electric and magnetic surface currents"



Shadow: opaque $\vec{E}=0, \vec{B}=0$ $\vec{E}_{sc} = -\vec{E}_0$

$\vec{B}_{sc} = -\vec{B}_0$

illuminated ideal conductor.

$\vec{E}_{\parallel}|_S = 0.$ $\vec{E}_{\parallel sc} = -\vec{E}_0$

$\vec{B}_{\perp}|_S = 0 \rightarrow \vec{B}_{\parallel, sc} = +\vec{B}_0$

"shadow" result is universal.

$$\vec{E}_0 = E_0 \vec{\epsilon}_0 \, e^{i(\vec{k}_0 \cdot \vec{x} - \omega t)} \qquad (\hat{k}_0 = \hat{z})$$

$$c\vec{B}_0 = \hat{k}_0 \times \vec{E}_0$$

$$\vec{\epsilon}^{\,*} \cdot \vec{F}_{shadow} = -\frac{ik E_0}{4\pi} \int da' \, \vec{\epsilon}^{\,*} \cdot \, e^{i\vec{k}_0 \cdot \vec{x}'} e^{-i\vec{k} \cdot \vec{x}'} \Big[\hat{n}' \times (\underbrace{\hat{k}_0 \times \vec{\epsilon}_0}_{(c\vec{B}_{sc})}) + \hat{k} \times (\underbrace{\hat{n}' \times \vec{\epsilon}_0}_{(\vec{E}_{sc})}) \Big]$$