3/25/2018

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = x$$

$$x \cdot y = \eta_{\mu\nu} \cancel{x}^\mu y^\nu = (x^0)(y^0) - (x^1)(y^1) - \dots$$

$$\boxed{\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}$$

Signature $+ - - -$

$$x \cdot x = c^2 t^2 - |\vec{x}|^2$$ → speed of light.

$$\boxed{x_\mu = \eta_{\mu\nu} x^\nu}$$ η maps (vector) → (covector)

$$x^\mu = (\eta^{-1})^{\mu\nu} x_\nu \qquad \boxed{\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}}$$

$$\boxed{\eta^{\mu\alpha} \eta_{\alpha\nu} = (\eta^{-1} \cdot \eta)^\mu{}_\nu = \delta^\mu{}_\nu} \qquad \delta^\mu{}_\nu = \eta^\mu{}_\nu$$

Λ $\quad x' = \Lambda x \quad \boxed{x'^\alpha = \Lambda^\alpha{}_\mu x^\mu} \quad \Lambda = 1 + \varepsilon$

$$x' \cdot y' = x \cdot y \to \boxed{\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0}$$

6 · D.f

Basis:

$$(J_i)_{jk} = \varepsilon_{ijk}$$

$$(J_1)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (J_2)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (J_3)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(K_1)_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix} \quad (K_2)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ -1 & & & \\ 0 & & & \end{pmatrix} \quad (K_3)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & & \\ -1 & & & \end{pmatrix}$$

Transformation: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$J^\mu{}_\nu = \eta^{\mu\alpha} J_{\alpha\nu} \qquad \Lambda^\mu{}_\nu = \eta^{\mu\alpha} \Lambda_{\alpha\nu}$$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Top row fixed
other three change sign.

$$(J_1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix} \quad (J_2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (J_3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix} \quad (K_2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix} \quad (K_3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & & \\ 1 & & & \end{pmatrix}$$

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + (\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})^\mu{}_\nu$$

$$\to \Lambda^\mu{}_\nu = \lim_{N\to\infty}\left(\delta^\mu + \frac{1}{N}(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})^\mu{}_\nu\right) = \exp\left[\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right]$$

J's $\rightarrow$ rotations $\quad \exp(\theta J_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$

K's $\rightarrow$ "boosts" $\quad \exp(\zeta K_1) = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

?? $\quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

$$ct' = \cosh\zeta \cdot ct + \sinh\zeta \cdot x$$
$$x' = \sinh\zeta \cdot ct + \cosh\zeta \cdot x$$

$x=0$
$$\left.\begin{aligned} x' &= \sinh\zeta \cdot ct \\ ct' &= \cosh\zeta \cdot ct \end{aligned}\right\} \quad x' = (c\tanh\zeta)\cdot t'$$

$$\boxed{x' = vt'}$$

point at rest in O $\rightarrow$ moving at speed v in O'

$\boxed{\frac{v}{c} = \tanh\zeta}$ $\quad \cosh^2\zeta - \sinh^2\zeta = 1$

$$1 - \tanh^2\zeta = \frac{1}{\cosh^2\zeta}$$

$$\cosh\zeta = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$ct' = \gamma(ct + \tfrac{v}{c}x)$$
$$x' = \gamma(\tfrac{v}{c}\cdot ct + x)$$

$$\boxed{\begin{aligned} x' &= \gamma(x + vt) \\ t' &= \gamma(t + \tfrac{xv}{c^2}) \end{aligned}}$$

$$\Lambda^T \eta \Lambda = \eta \rightarrow (\det\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$$

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu} \qquad \mu=\nu=0$$

$$\eta_{\alpha\beta}\Lambda^\alpha{}_0 \Lambda^\beta{}_0 = \eta_{00}(\Lambda^0{}_0)^2 + \sum_i \underset{-1}{\eta_{ii}} (\Lambda^i{}_0)^2 = \eta_{00} = 1$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2$$

$$\Lambda^0{}_0 \geq 1$$
$$\Lambda^0{}_0 \leq -1$$

$$\det\left(\exp(\vec\theta\cdot\vec J + \vec\zeta\cdot\vec K)\right)$$

$$\det M = \lambda_1 \cdots \lambda_N \qquad \log\det M = \log\lambda_1 + \dots + \log\lambda_N = \mathrm{Tr}(\ln M)$$

$$\det M = \exp(\mathrm{Tr}(\log M))$$

$$\det\Lambda = \exp\left(\mathrm{Tr}(\vec\theta\cdot\vec J + \vec\zeta\cdot\vec K)\right) = e^0 = 1.$$

$\exp(\vec\theta\cdot\vec J + \vec\zeta\cdot\vec K)$ continuously connected to $\mathbb{1}$

$\rightarrow \det = 1 \qquad \Lambda^0{}_0 \geq 1 \qquad (\cosh\zeta \geq 1)$

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \begin{matrix}\Lambda^0{}_0 \leq -1 \\ \det\Lambda = -1\end{matrix} \qquad P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \begin{matrix}\Lambda^0{}_0 \geq 1 \\ \det\Lambda = -1\end{matrix}$$

time reversal. $\qquad\qquad$ parity.

$$PT = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \begin{matrix}\det = +1 \\ \Lambda^0{}_0 \leq -1\end{matrix}$$

Four disconnected components, $\underline{\Lambda}$ $\underline{P\Lambda}$ $\underline{T\Lambda}$ $\underline{PT\Lambda}$

$$\Lambda^T \eta \Lambda = \eta$$

$$\eta^{-1} \Lambda^T \eta \Lambda = 1 \qquad \underline{\eta^{-1} \Lambda^T \eta} = \Lambda^{-1}$$

$$(\Lambda^{-1})^{\alpha}{}_{\beta} = (\eta^{-1})^{\alpha\mu} (\Lambda^T)_{\mu}{}^{\nu} \eta_{\nu\beta}$$

$$= \cancel{\eta^{\alpha\mu} \Lambda_{\nu}{}^{\mu} \eta_{\nu\beta}} =$$

$$= \eta_{\nu\beta} \Lambda^{\nu}{}_{\mu} \eta^{\alpha\mu} = \underline{\underline{\Lambda_{\beta}{}^{\alpha}}}$$

rotation $\eta^2 = 1$. $R^{-1} = R^T$. $(\theta \to -\theta)$

Boost $\underline{\Lambda^T = \Lambda}$ but $\eta \Lambda \eta \to \left(\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right)$ $(v \to -v)$

$\alpha\,(-)^2$ $\xi \to -\xi$