3/30/2016

$x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$

$d\tau^2 = dt^2 - \frac{1}{c^2} d\vec{x}^2 = \frac{1}{c^2} dx \cdot dx = \left(\frac{dt}{\gamma}\right)^2$

$u^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} u^0 \\ \vec{u} \end{pmatrix} = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix}$ 4-velocity.

$\boxed{u \cdot u = c^2}$ timelike (>0)

$a^\mu = \frac{du^\mu}{d\tau}$

$\frac{d}{d\tau}(u \cdot u) = 2u \cdot a = 0$ $\boxed{u \cdot a = 0}$

at rest. $a^\mu = \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$ $\boxed{a \cdot a = -|\vec{a}|^2}$

spacelike

$$a^\mu = \frac{du^\mu}{d\tau}$$

$$u \cdot u = c^2$$

$$\frac{d}{d\tau}(u \cdot u) = 2u \cdot a = 0 \qquad \boxed{u \cdot a = 0}$$

at rest (proper). $\vec{v} = 0$ $\quad u = \begin{pmatrix} c \\ \vec{0} \end{pmatrix}$

$$a = \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$$

$$a \cdot a = -|\vec{a}|^2_{\text{proper}}$$

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constant proper acceleration $|\vec{a}|_{\text{proper}} = g$.

$$u \cdot u = (u^t)^2 - (u^x)^2 = c^2$$

$$a \cdot a = (a^t)^2 - (a^x)^2 = -g^2$$

$$u \cdot a = (u^t)(a^t) - (u^x)(a^x) = 0$$

remove x-components $\qquad (a^x)^2 = (a^t)^2 + g^2 = a^2 + g^2$

$$(u^x)^2 = (u^t)^2 - c^2 = u^2 - c^2$$

$$u^2 a^2 = (u^t)^2 (a^t)^2 = (u^x)^2 (a^x)^2 = (u^2 - c^2)(a^2 + g^2)$$

$$\cancel{u^2 a^2} = \cancel{u^2 a^2} + u^2 g^2 - a^2 c^2 - g^2 c^2$$

$$\boxed{\frac{u^2}{c^2} - \frac{a^2}{g^2} = 1}$$

3

$$\boxed{\left(\frac{du}{d\tau}\right)^2 = g^2\left(\frac{u^2}{c^2}-1\right)}$$

$$\int_c^u \frac{du'}{\sqrt{\frac{u'^2}{c^2}-1}} = c\cdot\cosh^{-1}\left(\frac{u}{c}\right)\Big|_c^u = c\cdot\cosh^{-1}\frac{u}{c} = g\tau$$

$$\boxed{u = c\cdot\cosh\left(\frac{g\tau}{c}\right)} = c\frac{dt}{d\tau}$$

$$\boxed{ct = \frac{c^2}{g}\sinh\left(\frac{g\tau}{c}\right)} \qquad ct \approx \frac{c^2}{\cancel{g}}\frac{\cancel{g}\tau}{\cancel{c}} = c\tau$$

$$\left(\frac{dx}{d\tau}\right)^2 = \cancel{c^2 - \left(\frac{dt}{d\tau}\right)^2} \qquad u^2 - c^2 = c^2\left(1-\cosh^2\frac{g\tau}{c}\right)$$

$$= c^2\cdot\sinh\frac{g\tau}{c}$$

$$\boxed{\frac{dx}{d\tau} = c\cdot\sinh\frac{g\tau}{c}}$$

$$\boxed{x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right)-1\right]} \qquad x \approx \frac{c^2}{g}\left(\cancel{1}+\frac{1}{2}\left(\frac{g\tau}{c}\right)^2-\cancel{1}\right) = \frac{1}{2}g\tau^2$$

$$\left(x+\frac{c^2}{g}\right)^2 - c^2t^2 = \frac{c^2}{g}\left(\cosh^2\frac{g\tau}{c} - \sinh^2\frac{g\tau}{c}\right) = \frac{c^2}{g}$$

hyperbolic motion

$\frac{60\ \text{mph}}{2.7\ \text{sec}} = \frac{88\ \text{ft/sec}}{2.7\ \text{sec}} = 32.6\ \text{ft/s}^2$

Another way. (~~[illegible]~~)

$$u = \Lambda u_0 = \exp(\zeta K)\, u_0 \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\left.\begin{aligned} (ct)' &= \gamma\left(ct + \frac{v}{c}x\right) \\ x' &= \gamma\left(x + \frac{v}{c}\cdot ct\right) \end{aligned}\right\} \begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\frac{du}{d\tau} = \dot{\zeta} K \cdot \exp(\zeta K)\cdot u_0 = a = \exp(\zeta K)\, a_0 =$$

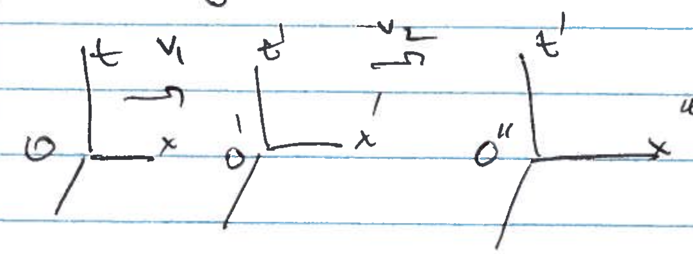
$$@\,\zeta = 0 \qquad \dot{\zeta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} = \dot{\zeta}\begin{pmatrix} 0 \\ c \end{pmatrix}$$

$$\dot{\zeta} = g/c$$

$$u = \exp\left(\frac{g\tau}{c} K\right)$$

$$\boxed{\begin{aligned} u^t &= c \cdot \cosh\left(\frac{g\tau}{c}\right) \\ u^x &= c \cdot \sinh\left(\frac{g\tau}{c}\right) \end{aligned}}$$

velocity composition



$$x' = \Lambda_1 x \qquad x'' = \Lambda_2 x'$$

$$\boxed{x'' = \Lambda x}$$

$$\Lambda = \Lambda_2 \Lambda_1 = \begin{pmatrix} \gamma_2 & \gamma_2 \frac{v_2}{c} \\ \frac{\gamma_2 v_2}{c} & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1 \frac{v_1}{c} \\ \gamma_1 \frac{v_1}{c} & \gamma_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right) & \gamma_1\gamma_2\left(\frac{v_1}{c} + \frac{v_2}{c}\right) \\ \gamma_1\gamma_2\left(\frac{v_1}{c} + \frac{v_2}{c}\right) & \gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \gamma \frac{v}{c} \\ \gamma \frac{v}{c} & \gamma \end{pmatrix}$$

$$\frac{\frac{\gamma V}{c}}{\gamma} = \frac{\cancel{\gamma_1}\cancel{\gamma_2}\left(\frac{v_1}{c} + \frac{v_2}{c}\right)}{\cancel{\gamma_1}\cancel{\gamma_2}\left(1 + \frac{v_1 v_2}{c^2}\right)}$$

$$\boxed{V = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}}$$

$$\exp(\rho_1 K)\exp(\rho_2 K)$$

$$= \exp\left((\rho_1 + \rho_2) K\right)$$

$$\frac{V}{c} = \tanh(\rho_1 + \rho_2) = \frac{\tanh \rho_1 + \tanh \rho_2}{1 + \tanh \rho_1 \tanh \rho_2}$$