4/6/2016

$J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$ $\qquad \partial_\mu J^\mu = 0$ continuity (11.129)

$A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}$. $\qquad \partial_\mu A^\mu = 0$ Lorentz gauge. (11.131)

$$\boxed{\Box^2 A^\mu = \frac{4\pi}{c} J^\mu}$$ (11.133)

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$$\begin{pmatrix} 0 & -E & -E & -E \\ E & 0 & -B & B \\ E & B & 0 & -B \\ E & -B & B & 0 \end{pmatrix}$$ (11.136) (11.137)

$$\boxed{{}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}}$$

$\vec{E} \to \vec{B} \qquad \vec{B} \to -\vec{E}$ (11.140)

Maxwell (inhomog.) $\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$ (11.144)

Homogeneous: $\boxed{\partial_\alpha {}^*F^{\alpha\beta} = 0}$ (11.145)

Homogeneous equations ($\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$)

$$\boxed{\nabla_\alpha \left({}^*F^{\alpha\beta} \right) = 0} \quad (11.145)$$

$$\nabla_\alpha \left[\tfrac{1}{2} \varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \right) = \tfrac{1}{2} \varepsilon^{\alpha\beta\mu\nu} \left(\nabla_\alpha F_{\mu\nu} \right) = 0$$

$$\varepsilon^{\mu\alpha\beta\gamma} \left(\nabla_\alpha F_{\beta\gamma} \right) = 0$$

$$\boxed{\nabla_{[\alpha} F_{\beta\gamma]} = 0}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta}$$

$$- \nabla_\alpha F_{\gamma\beta} - \nabla_\beta F_{\alpha\gamma} - \nabla_\gamma F_{\beta\alpha} = 0$$

$$\boxed{\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0} \quad (11.143)$$

Lorentz Force: $\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

$$\gamma \frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau} = q\left(\gamma\vec{E} + \frac{\gamma\vec{v}}{c} \times \vec{B}\right) \leftarrow F^{\alpha\beta}u_\beta = F^\alpha{}_\beta u^\beta$$

$\alpha = 1$: $$F^{1\beta}u_\beta = F^{10}u_0 + \cancel{F^{11}}u_1 + F^{12}u_2 + F^{13}u^3$$

$$= (E^x)(\gamma c) + (-B^z)(-\gamma v^y) + (+B^y)(-\gamma v^z)$$

$$= \gamma c\left(E^x + \frac{v^y}{c}B^z - \frac{v^z}{c}B^y\right) = \gamma c\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)^x$$

$$\frac{dp^i}{d\tau} = \frac{q}{c}F^{i\beta}u_\beta \quad (11.125).$$

$\alpha = 0$: $$\cancel{F^{00}}u_0 + F^{01}u_1 + F^{02}u_2 + F^{03}u_3 = (-\vec{E}) \cdot \left(-\frac{\vec{v}}{c}\right)$$

$$\gamma \frac{dE}{dt} = \gamma(\vec{F} \cdot \vec{v}) = \gamma q \vec{E} \cdot \vec{v}$$

$$\frac{d}{d\tau}\left(\frac{E}{c}\right) = \frac{q}{c}\gamma\vec{v} \cdot \vec{E} \qquad \frac{dp^0}{d\tau} = \frac{q}{c}F^{0\beta}u_\beta$$

$$\boxed{\frac{dp^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}u_\beta} \quad (11.126)$$

$$\frac{dp^\alpha}{d\tau} = m\frac{du^\alpha}{d\tau} = ma^\alpha \qquad u_\alpha a^\alpha = u \cdot a = \frac{q}{mc}F^{\alpha\beta}u_\beta u_\alpha \nearrow 0,$$

④

Something we didn't know before:

$$F'^{\alpha'\beta'} = \Lambda^{\alpha'}{}_{\mu}\Lambda^{\beta'}{}_{\nu}F^{\mu\nu} = (\Lambda^{\alpha'}{}_{\mu})(F^{\mu\nu})(\Lambda^{T}{}^{\nu}{}_{\beta'})$$

$$\boxed{F' = \Lambda F \Lambda^T}$$ (natural order).

Boost in x-direction

$$F' = \begin{pmatrix} \gamma & \frac{\gamma v}{c} & & \\ \frac{\gamma v}{c} & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}\begin{pmatrix} \gamma & \frac{\gamma v}{c} & & \\ \frac{\gamma v}{c} & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\gamma^2(1-\frac{v^2}{c^2})E^x & -\gamma E^y - \frac{\gamma v}{c}B^z & -\gamma E^z + \frac{\gamma v}{c}B^y \\ \gamma^2(1-\frac{v^2}{c^2})E^x & 0 & -\gamma B^z - \frac{\gamma v}{c}E^y & \gamma B^y - \frac{\gamma v}{c}E^z \\ \gamma E^y + \frac{\gamma v}{c}B^z & \gamma B^z + \frac{\gamma v}{c}E^y & 0 & -B^x \\ \gamma E^z - \frac{\gamma v}{c}B^y & -\gamma B^y + \frac{\gamma v}{c}E^x & B^x & 0 \end{pmatrix}$$

$B'^x = B^x$ $E'^x = E^x$ $E'_\parallel = E_\parallel$ $B'_\parallel = B_\parallel$

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \frac{\vec{v}}{c}\times\vec{B}_\perp)$$

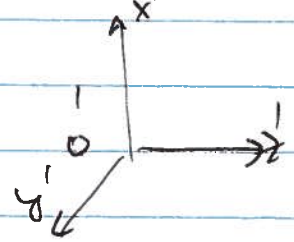
$$\vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{\vec{v}}{c}\times\vec{E}_\perp)$$

⟸ ~~Newtonian~~ $F' = F$

(11.148)

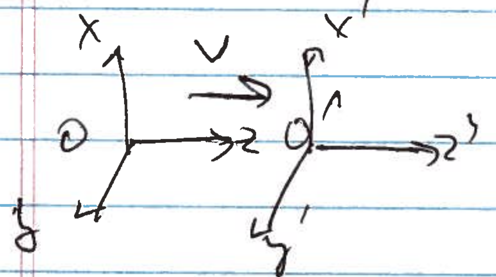
(11.149)

point q



@ rest. $\vec{E}' = \frac{q\hat{r}'}{r'^2} = \frac{q\vec{x}'}{(r'^2)^{3/2}}$

$$= q\left(\frac{x'\hat{x}' + y'\hat{y}' + z'\hat{z}'}{(x'^2+y'^2+z'^2)^{3/2}}\right)$$



$x = x'$ $\quad$ $y = y'$

$$z = \gamma(z' + vt')$$
$$t = \gamma\left(t' + \frac{z'v}{c^2}\right)$$

Origins coincide @ $t = t' = 0$

$$E^z = E_\parallel = E'_\parallel = E'^{z'} = \frac{qz'}{(x'^2+y'^2+z'^2)^{3/2}} = q\frac{\gamma(z-vt)}{(x^2+y^2+\gamma^2(z-vt)^2)^{3/2}}$$

$$\vec{E}_\perp = \gamma\left(\vec{E}'_\perp + \cancel{\frac{\vec{v}}{c}\times\vec{B}_\perp}\right) = \gamma\vec{E}'_\perp = \frac{\gamma\vec{x}'_\perp}{(r')^3}$$

$$E^x = \frac{\gamma q x}{(x^2+y^2+(z-vt)^2)^{3/2}} \qquad (y').$$

@ $t=0$.

$$\vec{E} = \frac{\gamma q(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2+y^2+\gamma^2 z^2)^{3/2}}$$

$z = r\cos\theta$ etc.

$$\vec{E} = \frac{\gamma q\vec{x}}{r^3(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \gamma^2\cos^2\theta)^{3/2}}$$

$$\vec{E} = q\frac{\gamma\hat{r}}{r^2}\frac{1}{(\sin^2\theta + \gamma^2\cos^2\theta)^{3/2}}$$

$$= q\frac{\hat{r}}{r^3}\frac{1}{\gamma^2}\frac{1}{\left((1-v^2/c^2)\sin^2\theta + \cos^2\theta\right)^{3/2}}$$

$$\boxed{\vec{E} = \frac{q\hat{r}}{\gamma^2 r^2}\frac{1}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}}$$

radial

inverse square

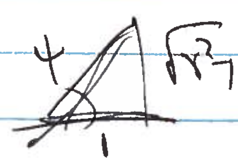
$$\oint \vec{E}\cdot\hat{r}\, da = 4\pi q$$

$$\int d\Omega \frac{1}{\gamma^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} = \int d\Omega \frac{\gamma}{\left(1+(\gamma^2-1)\cos^2\theta\right)^{3/2}}$$

$$= 2\pi\int_{-1}^{1} d\mu \frac{\gamma}{\left(1+(\gamma^2-1)\mu^2\right)^{3/2}}.$$

$$(\gamma^2-1)\mu^2 = \tan^2\psi \rightarrow 2\pi\gamma\int_{-\psi_0}^{\psi_0}\frac{\sec^2\psi\, d\psi}{\sqrt{\gamma^2-1}}\frac{1}{(1+\tan^2\psi)^{3/2}}$$

$$\sqrt{\gamma^2-1}\, d\mu = \sec^2\psi\, d\psi.$$



$$= \frac{2\pi\gamma}{\sqrt{1-\gamma^2}}\cdot 2\int_0^{\psi_0}\cos\psi\, d\psi$$

$$\left(\sin\psi\right]_0^{\psi_0} = \sin\psi_0$$

$$= \frac{\sqrt{\gamma^2-1}}{\gamma}$$

$$= 4\pi$$