4/18/2016 (§12.7)

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \qquad \phi_j \quad (A_\mu)$$

$\delta S = 0 \rightarrow$

$$\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0} \quad (12.83)$$

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\vec{\nabla}\phi|^2 - V(\phi) = \frac{1}{2}(\partial_\alpha \phi)(\partial_\beta \phi)\eta^{\alpha\beta} - V$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\partial_\mu(\partial^\mu \phi) = \Box^2 \phi = -\frac{\partial V}{\partial \phi}$$

$V = \frac{1}{2} m^2 \phi^2 \qquad \Box^2 \phi = -m^2 \phi \qquad$ Klein Gordon

$$e^{i(\vec{k}\cdot\vec{x} - \omega t)} \longrightarrow -\omega^2 + |\vec{k}|^2 = m^2$$

$$(\hbar\omega)^2 = (\hbar c \vec{k})^2 + (mc^2)^2$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \pi^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\vec{\nabla}\phi|^2 + V\right)$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + V$$

energy density $\qquad$ kinetic + strain + potential

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$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

$$\boxed{T^{\mu\nu} = \phi^{,\mu} \phi^{,\nu} - \eta^{\mu\nu} \mathcal{L}}$$

$u = \varepsilon.\ \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + V$

$\vec{S} : T^{0i} = \dot{\phi}\phi^{,i} = -\dot{\phi}\phi_{,i}$

$$T^{00} = \rho \qquad T^{0i} = \vec{S} \qquad T^{i0} = \vec{g} \qquad T^{ij} = (\text{pressure})$$

$$\partial_\mu (T^{\mu\nu}) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi \right) - \eta^{\mu\nu} \partial_\mu \mathcal{L}$$

$$= \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right) \partial^\nu \phi + \partial_\mu (\partial^\nu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

$$- \eta^{\mu\nu} \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} \partial_\mu (\partial_\sigma \phi) \right)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu (\partial_\mu \phi) \quad = 0$$

$$- \frac{\partial \mathcal{L}}{\partial \phi} \partial^\nu \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} \partial^\nu (\partial_\sigma \phi)$$

$$\frac{\partial \varepsilon}{\partial t} + \vec{\nabla} \cdot \vec{S} = \frac{\partial}{\partial t}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\vec{\nabla}\phi|^2\right) + \vec{\nabla} \cdot (-\dot{\phi}\vec{\nabla}\phi)$$

$$= \dot{\phi}\ddot{\phi} + \vec{\nabla}\phi \cdot \vec{\nabla}\dot{\phi} - \vec{\nabla}\dot{\phi} \cdot \vec{\nabla}\phi - \dot{\phi}\nabla^2\phi$$

E&M Vector field A^μ

"KE" $\dot{A}^2 \to \partial_\mu A_\nu \to F_{\mu\nu}$ gauge invariant

Scalar $\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}}$ (12.55).

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F_{\rho\sigma} \eta^{\rho\alpha} \eta^{\sigma\beta}$$

$$= -\frac{1}{16\pi} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \eta^{\rho\alpha} \eta^{\sigma\beta}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{16\pi} \Big(\alpha = \mu, \beta = \nu \quad + \quad \beta = \mu, \alpha = \nu$$
$$+ \rho = \mu, \sigma = \nu \quad + \quad \sigma = \mu, \rho = \nu \Big)$$

$$= -\frac{1}{16\pi} \Big(F_{\rho\sigma} \eta^{\rho\mu} \eta^{\sigma\nu} - F_{\rho\sigma} \eta^{\rho\nu} \eta^{\sigma\mu}$$
$$+ F_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta} - F_{\alpha\beta} \eta^{\nu\alpha} \eta^{\mu\beta} \Big)$$

$$= -\frac{1}{16\pi} \left(F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \right)$$

$$= -\frac{1}{4\pi} F^{\mu\nu} = \cancel{+\frac{1}{4\pi} F^{\nu\mu}}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\partial_\mu \left(\frac{1}{4\pi} F^{\mu\nu} \right)$$

$$= \frac{\partial \mathcal{L}}{\partial(A_\nu)} = \cancel{\;}$$

$$\partial_\mu \left(F^{\mu\nu} \right) = -4\pi \cdot \frac{\partial \mathcal{L}}{\partial(A_\nu)} = \frac{4\pi}{c} J^\nu$$

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu} \quad (12.85)$$

$A_\mu J^\mu \rightarrow$ source for A_μ, also implies interaction of matter with E/M

<u>Weak</u> interactions → vector field (W_μ)

<u>massive</u> (?) $V(A) \rightarrow \frac{1}{2} m^2 A^2 = \frac{1}{2} A_\alpha A_\beta \eta^{\alpha\beta}$

Not gauge invariant. — SSB

Symmetries: $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})\cdot(\partial^\mu \vec{\phi}) - V(\vec{\phi}\cdot\vec{\phi})$$

$$= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1^2 + \phi_2^2).$$

$$\vec{\phi}' = R_\alpha \vec{\phi} \qquad \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$\frac{d\mathcal{L}}{d\alpha} = 0$ symmetry.

Let $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\phi})}\cdot\frac{d\vec{\phi}}{d\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\frac{d\phi_k}{d\alpha}$

$$\partial_\mu J^\mu = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\frac{d\phi_k}{d\alpha}\right)$$

$$= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\left(\frac{d\phi_k}{d\alpha}\right) + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right)\left(\partial_\mu\left(\frac{d\phi_k}{d\alpha}\right)\right)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_k}\frac{d\phi_k}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\frac{d}{d\alpha}(\partial_\mu \phi_k)$$

$$= \frac{d\mathcal{L}}{d\alpha} = 0$$

Symmetry $\rightarrow$ Conserved Current

Noether's Theorem, most important result of 20th Century

Broken symmetry. $V = \frac{1}{4}\lambda\left(\vec{\phi}\cdot\vec{\phi} - a^2\right)^2$

$\mathcal{L} = \frac{1}{2}\,\partial\vec{\phi}\cdot\partial\vec{\phi} - V$

minimum energy configuration:

$\vec{\phi}$ = constant, $|\vec{\phi}| = a$, $\partial\phi = 0$

$\vec{\phi} \in \mathcal{M}_0 = \{V(\vec{\phi}) = 0\}$.

$\phi_2 = a$ $\quad$ $\phi_1 = 0$

$\vec{\phi} = \vec{\phi}_0 + \delta\vec{\phi} = \begin{pmatrix} 0 \\ a \end{pmatrix} + \begin{pmatrix} \delta\phi_1 \\ \delta\phi_2 \end{pmatrix}$

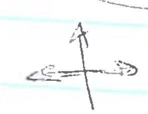
$\cancel{a^2} + 2a\,\delta\phi_2 + \delta\phi_2^2$

$\mathcal{L} = \frac{1}{2}(\partial\delta\vec{\phi})\cdot(\partial\delta\vec{\phi}) - \frac{1}{4}\lambda\left[\delta\phi_1^2 + (a+\delta\phi_2)^2 - \cancel{a^2}\right]^2$

$= \cdots - \frac{1}{4}\Big(\underbrace{4a^2\,\delta\phi_2^2}_{\phi^2} + \underbrace{4a\,\delta\phi_2(\delta\phi_1^2 + \delta\phi_2^2)}_{\phi^3} + \underbrace{(\delta\phi)^4}_{\phi^4}\Big)$

order $\delta\phi^2$ $\quad$ $\mathcal{L} = \frac{1}{2}\,\partial\delta\vec{\phi}\cdot\partial\delta\vec{\phi} - \frac{\lambda}{2}(2a^2)\,\delta\phi_2^2 + \dots$

massless $\delta\phi_1$ $\quad$ $\boxed{m_2^2 = 2\lambda a^2}$



Goldstone's Theorem