4/20/2010 Another theory with a symmetry.
(sort of the same).

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi^* \phi) \qquad \phi = \phi_1 + i\phi_2$$
$$\phi^* = \phi_1 - i\phi_2$$

$\phi \to e^{i\alpha}\phi$. $\phi^* \to e^{-i\alpha}\phi$. $|\phi|^2, \mathcal{L}$ unchanged

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{d\phi}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{d\phi^*}{d\alpha}$$

$$= \frac{1}{2}(\partial^\mu \phi^*)(i\phi) + \frac{1}{2}(\partial^\mu \phi)(-i\phi^*)$$

$$\boxed{J^\mu = \frac{i}{2}(\partial^\mu \phi^* \cdot \phi - \phi^* \cdot \partial^\mu \phi)}$$

Suppose $\alpha \to \alpha(x)$. $|\phi|^2$ ✓

$$\partial_\mu(e^{i\alpha}\phi) = e^{i\alpha}(\partial_\mu \phi) + i(\partial_\mu \alpha)\phi.$$

gradient of α appears.

recall: gauge transformation
shifts A_μ by gradient

②

ϕ, A_μ (def) $D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$

gauge transformation. $\phi \to e^{i\alpha}\phi$

$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$

leaves $\phi^*\phi$, $F_{\mu\nu}$ invariant

$$D'_\mu \phi' = \left(\partial_\mu + ie\left(A_\mu - \frac{1}{e}\partial_\mu \alpha\right)\right)(e^{i\alpha}\phi)$$

$$= e^{i\alpha}\partial_\mu\phi + \cancel{i\partial_\mu\alpha\, e^{i\alpha}\phi}$$

$$+ ie\overset{e^{i\alpha}}{A_\mu}\phi + \cancel{ie\left(-\frac{1}{e}\partial_\mu\alpha\right)e^{i\alpha}\phi}$$

$$= e^{i\alpha}(D_\mu\phi)$$ covariant derivative

$$\boxed{\mathcal{L} = \frac{1}{2}(D\phi)^*\cdot(D\phi) - V(\phi^*\phi)}$$ invariant

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi + ieA_\mu)^*(\partial^\mu\phi + ieA_\mu)$$

$$= \frac{1}{2}(\partial\phi^*\cdot\partial\phi - ieA\phi^*\cdot\partial\phi + \partial\phi^* \cdot ieA\phi + e^2 A\cdot A\,\phi^*\phi$$

$$= \frac{1}{2}(\partial\phi)^*(\partial\phi) + \frac{1}{2}e^2 A\cdot A\,\phi^*\phi - V(\phi)$$

$$+ A_\mu\left(\frac{1}{2}ie\,\partial^\mu\phi^*\cdot\phi - \frac{1}{2}ie\,\phi^*\cdot\partial^\mu\phi\right)$$

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$$J^\mu = \frac{\partial \mathcal{L}}{\partial A_\mu} = e \; \frac{i}{2}\left(\partial^\mu \phi^* \cdot \phi - \phi^* \cdot \partial^\mu \phi \right)$$

Gauge invariance $\longleftrightarrow$ conservation of charge.

Non-Abelian Theory $\qquad [T_a, T_b] = C^c{}_{ab} T_c$.

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \qquad \vec{W}_\mu = \begin{pmatrix} W_{1,\mu} \\ W_{2,\mu} \\ W_{3,\mu} \end{pmatrix}$$

$$\mathcal{D}_\mu \vec{\phi} = \partial_\mu \vec{\phi} - e \vec{W}_\mu \wedge \vec{\phi} \qquad (\text{no } i\text{s.})$$

$$\phi' = R\phi \qquad R = \exp(\vec{\alpha} \cdot \vec{J}) \qquad [J_i, J_j] = \varepsilon_{ijk} J_k$$

$$\underset{\sim}{W}_\mu = \vec{W}_\mu \cdot \vec{J}$$

$$\underset{\sim}{W}' = R W R^{-1} + \frac{1}{e} (\partial_\mu R) R^{-1}$$

(small α, $\quad \vec{W}_\mu' = \vec{W}_\mu + \vec{\alpha} \wedge \vec{W}_\mu + \frac{1}{e} \partial_\mu \vec{\alpha}$).

$$\vec{G}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - e \vec{W}_\mu \wedge \vec{W}_\nu$$

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] \vec{\phi} = e \underset{\sim}{G}_{\mu\nu} \vec{\phi}$$

$$\mathcal{L} = \underset{(\frac{1}{2})}{(D_\mu \vec{\phi})\cdot(D^\mu \vec{\phi})} - \frac{1}{4}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi}-a^2)^2$$

<u>ground state</u> $\vec{\phi} = a\begin{pmatrix}0\\0\\1\end{pmatrix} = a\hat{n}$

$$D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} = (\partial\phi - e\vec{W}_\wedge\vec{\phi})\cdot(\partial - e\vec{W}_\wedge\vec{\phi})$$

$$= (\partial\phi)^2 + \underline{\vec{J}\cdot\vec{W}_\mu} + e^2(\vec{W}_\wedge\vec{\phi})\cdot(\vec{W}_\wedge\vec{\phi})$$

$$\to e^2\,\vec{W}\cdot(\vec{\phi}_\wedge(\vec{W}_\wedge\vec{\phi}))$$

$$= e^2\,\vec{W}\cdot\left[\vec{W}(\vec{\phi}\cdot\vec{\phi}) - \vec{\phi}(\vec{W}\cdot\vec{\phi})\right)$$

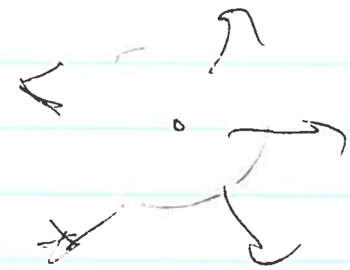
$$= e^2 a^2 (W^2 - (\hat{n}\cdot\vec{W})^2)$$

$$= e^2 a^2 (W_1^2 + W_2^2)$$

$m_1^2 = m_2^2 = e^2 a^2$ $\qquad m_3^2 = 0$ $\qquad$ rotations about $\hat{n}$ leave $\langle\vec{\phi}\rangle$ unchanged.

$$E = \int d^3x \left[\,|\dot{\vec{\phi}}|^2 + \frac{1}{2}|\dot{\vec{W}}|^2 + |D_i\vec{\phi}|^2 + \frac{1}{4}|\vec{G}_{ij}|^2 + V(|\vec{\phi}|)\right)$$

Solution $\vec\phi \xrightarrow[(r\to\infty)]{} a\hat r$



$D_\mu\vec\phi = \partial_\mu\vec\phi - e\,\vec W_\mu \wedge \vec\phi$

$\leftarrow \frac{1}{r}\frac{\partial\vec\phi}{\partial\theta} = O(1/r)$ $\quad \int d^3x\, |\partial\phi|^2 \to \infty$.

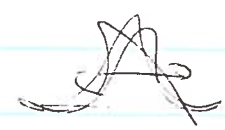
$W \propto \frac{1}{r}$.

$\vec\phi = \hat r\,\frac{H(aer)}{r}$ $\quad W_a^i = \epsilon_{aij}\hat r_j\left(1-\frac{K(aer)}{er}\right)$

$W_a^0 = 0$.

$\lambda\to 0$ $\quad K = \frac{\xi}{\sinh\xi}$ $\quad H = \xi\frac{\cosh\xi}{\sinh\xi} - 1$ $\quad \xi = aer$

exponentially cutoff energy



$\Delta\xi \sim 1$.

$\Delta r \simeq \frac{1}{ae} = \frac{1}{m_W}$

$\xi\to\infty$. $K\to 0$. $\quad W_a^i = \epsilon_{aij}\hat r_j\,\frac{1}{er}$

$F^{\mu\nu} = \hat\phi\cdot\vec G^{\mu\nu} \to F_{ij} = \epsilon_{ijk}\frac{\hat r_k}{er^2}$

$\vec B = g\,\frac{\hat r}{r^2}$ $\quad g = -\frac{4\pi}{e}$; Dirac $\frac{eg}{4\pi} = \frac{1}{2}n$

$q = \frac{1}{2}e$

Works $\underline{\text{whenever}}$ $\underline{\underline{G}} \longrightarrow (G') \times \underset{\text{E.M.}}{\underline{\underline{U(1)}}}$

Unification → charge quantization

Unification → monopoles