PHZ 6607 Fall 2016

End Term Exam, Due 6:00pm, Tuesday, December 13

1. A two-dimensional world described by the metric

$$ds^2 = a^2 \left(\frac{dx^2}{1+x^2} + x^2 \, d\theta^2 \right)$$

has more symmetry than you think.

(a) It is apparent that the metric does not depend on θ. Verify that $\hat{\boldsymbol{u}} = \hat{\boldsymbol{e}}_\theta$ is a Killing vector. Verify that p_θ is a conserved quantity.

(b) Show that the vector

$$\hat{\boldsymbol{v}} = \sqrt{1+x^2} \cos\theta \, \hat{\boldsymbol{e}}_x - \frac{\sqrt{1+x^2}}{x} \sin\theta \, \hat{\boldsymbol{e}}_\theta .$$

is a Killing vector.

(c) Find another Killing vector $\hat{\boldsymbol{w}}$ for this geometry.

2. Metrics g and g' are related by a conformal transformation, $g'_{\mu\nu} = e^{2\Omega} \, g_{\mu\nu}$.

(a) Show that if ξ^μ is a Killing vector for the metric g, then it is a conformal Killing vector for the metric g'; that is, it satisfies

$$\nabla'_{(\alpha} \xi'_{\beta)} = 2\xi^\mu \nabla_\mu \Omega \, g'_{\alpha\beta} .$$

(b) Show that if ξ^μ is a conformal Killing vector, then $k^\mu \xi_\mu$ is conserved along a null geodesic, where $k^\mu = dx^\mu / d\lambda$ is the geodesic tangent.

3. Consider gravity in the weak field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h| \ll 1$, including the effects of matter currents $T^{0i} = J^i$ as well as the mass density $T^{00} = \rho$. Let ϕ denote the solution to $\Box^2 \phi = 4\pi G \rho$ and $\boldsymbol{\zeta}$ the solution to $\Box^2 \boldsymbol{\zeta} = 16\pi G \boldsymbol{J}$.

(a) What are the metric perturbations $h_{\mu\nu}$ with both ρ and $\boldsymbol{J}$ as sources?

(b) Show that the geodesic equation leads to what looks like a "magnetic" gravitational force, and identify the magnetic field.

4. In weak gravity, the metric of a mass M at rest at the origin is

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\alpha\phi)\,\delta_{ij}\,dx^i dx^j\,,$$

where α is a constant (usually written as γ but given a different name for the domain of this problem for reasons that may become apparent) and $\phi = -[G]M/r$.

(a) What is the value of α in general relativity?

(b) Instead of sitting at rest at the origin, the mass M moves in the $+x$-direction with speed v, passing through the origin at time $t = 0$, so that its position as a function of time is $x = vt$. What is the metric in this case?

(c) A photon moves along a trajectory originally parallel to the y-axis with an offset b in the x-direction, so that its undeflected trajectory is $\boldsymbol{x}_0 = -b\,\hat{\boldsymbol{x}} + t\,\hat{\boldsymbol{y}}$. Assuming the angle is small, by what angle is the path of this test particle deflected?

(d) What is the change in the energy of the deflected photon in part (c)?

5. The star S0-2 orbits the $3.7 \times 10^6\,M_\odot$ black hole of Sagittarius A* with a period of 14.53 year in an orbit with eccentricity $e = 0.8670$.

(a) What is the ratio of the pericenter distance of the orbit to the Schwarzschild radius of the black hole?

(b) How fast does the orbital perihelion advance?

(c) Ghez et al. (2004) say, "S0-2's spectral features are consistent with those of an O8–B0 dwarf, suggesting that it is a massive ($M \sim 15\,M_\odot$), young ($< 10\,\mathrm{Myr}$), main-sequence star. What is the rate at which the system radiates energy in gravitational waves? Estimate the lifetime for coalescence.