PHZ 6607 Fall 2016

Homework #3, Due Friday, September 23

1. $T^{\alpha\beta\cdots\rho}$ is a tensor in an n-dimensional space.

(a) If T has rank r (r indices) and no symmetries, how many independent components does it have?

(b) If T is antisymmetric in a indices, $T^{\alpha\beta\cdots\rho} = T^{[\alpha\beta\cdots\rho]}$, how many independent components does it have? Test your expression for $n = 4$ and $a = 2$, $a = 3$, $a = 4$. (list the independent components explicitly).

(c) If T is symmetric in s indices, $T^{\alpha\beta\cdots\rho} = T^{(\alpha\beta\cdots\rho)}$, how many independent components does it have? Test your expression for $n = 4$ and $s = 2$, $s = 3$, $s = 4$.

(d) Let $A_{\mu\nu}$ be an antisymmetric tensor, $A_{\mu\nu} = A_{[\mu\nu]}$, let $S_{\mu\nu}$ be a symmetric tensor, $S_{\mu\nu} = S_{(\mu\nu)}$, and let $V_{\mu\nu}$ be arbitrary. Show that $V^{\mu\nu}A_{\mu\nu} = V^{[\mu\nu]}A_{\mu\nu}$. Show that $V^{\mu\nu}S_{\mu\nu} = V^{(\mu\nu)}S_{\mu\nu}$. What is $A_{\mu\nu}S^{\mu\nu}$?

2. For any timelike vector U^μ there is a frame in which U has only a t-component, and in this frame U is invariant under rotations, a three-dimensional subset of all possible Lorentz transformations. A null vector V^μ with components $V^t = V^z = 1$, $V^x = V^y = 0$ also has a three-dimensional subset of Lorentz transformations that leave the components of V unchanged, Wigner's "little group of V." A pure rotation in the x-y plane is one such transformation. Find a transformation that is not a pure rotation but is in the little group of V. Find another transformation that is not a pure rotation but is in the little group of V. Write the commutator algebra of the three generators of your two transformations and the x-y rotation mentioned above.

3. A Lorentz transformation is the product of a boost with rapidity ζ in direction "east", followed by a boost with rapidity ζ in direction "north", followed by a boost with rapidity ζ in direction "west", followed by a boost with rapidity ζ in direction "south", where ζ is the same in each. What is the resulting transformation? To lowest order for small ζ, is it a boost or a rotation? In which direction or about which axis? At what order does the other (boost or rotation) enter? In which direction or about which axis? What are the directions of resultant boosts and/or rotations when ζ is not small?

4. Observer $\mathcal{O}$ at rest sees electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$. Frame $\mathcal{O}'$ moves with speed v in the $+x$ direction with respect to $\mathcal{O}$.

(a) What are the fields $\boldsymbol{E}'$, $\boldsymbol{B}'$ in frame $\mathcal{O}'$?

(b) Write your answer to (a) in a way that does not involve the specific direction of v.

5. Observer $\mathcal{O}$ at rest sees a symmetric tensor $T^{\mu\nu}$ to be diagonal with components (ρ, p, p, p).

(a) What are the components of $T^{\mu}{}_{\nu}$? Of $T_{\mu}{}^{\nu}$? Of $T_{\mu\nu}$?

(b) Frame $\mathcal{O}'$ moves with speed v in the $+x$ direction with respect to $\mathcal{O}$. What are the components of $T^{\mu'\nu'}$ in frame $\mathcal{O}'$? What are the components of $T_{\mu'\nu'}$? How can the "rest frame" be identified? What is the restriction on p such that ρ is positive for all observers? What is the restriction on p such that $\rho - 3p$ is positive for all observers?