Lorentz Generators

We know from considering infinitesimal transformations that the Lorentz transformation generators must be antisymmetric, and so the only choice is an overall sign. Here they are with signs chosen for reasons given below. Let

$$M_{\mu\nu\alpha\beta} = -\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}.$$

This is an object for which the labels μ, ν tell which generator, and α, β label matrix row and column. The "natural" position of matrix indices is $(M_{\mu\nu})^{\alpha}{}_{\beta}$, where $(M_{\mu\nu})^{\alpha}{}_{\beta} = \eta^{\alpha\lambda}(M_{\mu\nu})_{\lambda\beta}$.

There are six independent generators, $M_{\mu\nu} = -M_{\nu\mu}$ (suppressing the matrix indices). Give them the alternative names

$$J_i = \tfrac{1}{2}\,\epsilon_{ijk}M_{jk},$$

for those with two space indices, and

$$K_i = M_{0i}$$

for those with one space and one time index. These are then explicitly

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The generators obey commutation relations

$$[J_i, J_j] = \epsilon_{ijk}J_k, \qquad [J_i, K_j] = \epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\epsilon_{ijk}J_k.$$

The first of these says that the J's generate rotations in three-dimensional space and fixes the overall sign of the J's.. The second says the K's transform as a vector under rotations, and the third completes the Lorentz group (and requires the same sign as the first for J)

A typical rotation has the form

$$\Lambda = \exp(\theta J_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

A typical boost has ($\cosh\zeta = \gamma$, $\tanh\zeta = v$).

$$\Lambda = \exp(\zeta K_1) = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$