8/26/2016 Lorentz transformation preserve.

$|\Delta\vec{x}|^2 = t^2 \qquad |\Delta\vec{x}'|^2 = t'^2.$

General $\boxed{x'^\mu = \Lambda^\mu{}_\alpha x^\alpha}$

$$x' \cdot y' = \eta_{\mu\nu}(\Lambda^\mu{}_\alpha x^\alpha)(\Lambda^\nu{}_\beta y^\beta) = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha y^\beta$$

$$= x \cdot y = \eta_{\alpha\beta} x^\alpha y^\beta \qquad \text{All } x, y \to$$

$$\boxed{\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}} \qquad \boxed{\Lambda^T \eta \Lambda = \eta}$$

$\Lambda^\mu{}_\alpha = \delta^\mu{}_\alpha$ works. $\left(\begin{array}{l} x' = x \\ y' = y \end{array} \to x' \cdot y' = x \cdot y\right)$

Look at $\Lambda^\mu{}_\alpha = \delta^\mu{}_\alpha + \varepsilon^\mu{}_\alpha$ (ε small).

$$\eta_{\mu\nu}(\delta^\mu{}_\alpha + \varepsilon^\mu{}_\alpha)(\delta^\nu{}_\beta + \varepsilon^\nu{}_\beta)$$

$\boxed{\eta_{\mu\nu} x^\mu = x_\nu}$

$$= \eta_{\alpha\beta} + \eta_{\alpha\nu}\varepsilon^\nu{}_\beta + \eta_{\mu\beta}\varepsilon^\mu{}_\alpha$$

$\boxed{\eta^{\mu\nu} x_\nu = x^\mu}$

$\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$

$$= \eta_{\alpha\beta} + \varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = \eta_{\alpha\beta}$$

$$\boxed{\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = 0} \qquad \text{Antisymmetric}$$

$\epsilon = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, six independent $\begin{Bmatrix} 01 & 02 & 03 \\ 12 & 13 & 23 \end{Bmatrix}$

Let $(M_{\mu\nu})_{\alpha\beta} = -\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}$

$(\mu\nu) \to$ which one. $(\alpha\beta)$: matrix indices.

$$(M_{01})_{\alpha\beta} = -\overset{\text{row } 0}{\eta_{0\alpha}}\,\underset{\text{col } 1}{\eta_{1\beta}} + \overset{\text{row } 1}{\eta_{1\alpha}}\,\underset{\text{col } 0}{\eta_{0\beta}} = \begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

"Natural" position of indices. $(M)^{\alpha}{}_{\beta} = \eta^{\alpha\nu} M_{\nu\beta}$

$$(M_{01})^{\alpha}{}_{\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & \\ & & & \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

same sign

$$(M_{12})_{\alpha\beta} = -\eta_{1\alpha}\eta_{2\beta} + \eta_{2\alpha}\eta_{1\beta} = \left(\begin{array}{c|ccc} & & & \\ \hline & & -1 & \\ & 1 & & \\ & & & \end{array}\right)$$

$$(M_{12})^{\alpha}{}_{\beta} = \left(\begin{array}{c|ccc} & & & \\ \hline & & -1 & \\ & 1 & & \\ & & & \end{array}\right)$$

opposite sign

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$M_{0i} = K_i$ $\quad \cancel{M_{ij} = J_k}$ $\quad J_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$

$$\Lambda = \mathbb{1} + \vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}$$

$$\Lambda^\alpha{}_\beta = \delta^\alpha_\beta + \theta_i (J_i)^\alpha{}_\beta + \zeta_i (K_i)^\alpha{}_\beta \qquad (\theta, \zeta \text{ small})$$

Finite $\rightarrow$ $\Lambda = \lim_{N\to\infty}\left(\mathbb{1} + \frac{1}{N}\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right)\right)^N$

$$= \exp\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right).$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad J_x,\ J_y,\ J_z$$

$$K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cosh\zeta & -\sinh\zeta \\ -\sinh\zeta & \cosh\zeta \end{pmatrix} \qquad K_x,\ K_y,\ K_z$$

$$t' = \cosh\zeta\, t - \sinh\zeta\, x = \cosh\zeta\,(t - \tanh\zeta \cdot x)$$
$$x' = \cosh\zeta\, x - \sinh\zeta \cdot t = \cosh\zeta\,(x - \tanh\zeta \cdot t)$$

x' depends on $(x - \tanh\zeta\, t) = (x - vt)$

$$v = \tanh\zeta$$

$\cosh^2 - \sinh^2 = 1$ $\qquad \cosh\zeta = \frac{1}{\sqrt{1-\tanh^2\zeta}} = \frac{1}{\sqrt{1-v^2}} = \gamma$

$1 - \tanh^2 = \frac{1}{\cosh^2}$

Properties

$$\eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

$$\to \eta_{\alpha\beta}\Lambda^\alpha{}_0\Lambda^\beta{}_0 = \eta_{00} = -1$$

$$= (\Lambda^0{}_0)^2(-1) + \sum_i(\Lambda^i{}_0)^2(+1)$$

$$\boxed{(\Lambda^0{}_0)^2 = 1 + \sum(\Lambda^i{}_0)^2}$$

$\Lambda^0{}_0 \geq 1$

$\Lambda^0{}_0 \leq -1$

$$\Lambda^T\eta\Lambda = \eta \to (\det\Lambda)^2 = 1 \qquad \boxed{\det\Lambda = \pm 1}$$

We have $\Lambda^0{}_0 = \cosh\zeta \geq 1$.

we have. $\log\det M = \log\left(\prod_i \lambda_i\right) = \sum \log\lambda_i = \mathrm{Tr}\log M$.

$$\mathrm{Tr}\,J = \mathrm{Tr}\,K = 0. \quad \mathrm{Tr}\log\Lambda = 0 \qquad \boxed{\det\Lambda = +1}$$

what we have is continuously connected to $\mathbb{1}$ $\to$ can't change discrete index

"Proper Λ".

define. $P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $T = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

parity time-reversal

$(Px)\cdot(Py) = x\cdot y$ $(Tx)\cdot(Ty) = x\cdot y.$

$\exp(\vec{\theta}\cdot\vec{J} + \vec{J}\cdot\vec{K})$ "proper" $\det\Lambda = 1$ $\Lambda^0_0 \geq 1$

$(\Lambda_p)\times P$ $\det = -1$ $\Lambda^0_0 \geq 1$

$(\Lambda_p)\times T$ $\det = -1$ $\Lambda^0_0 \leq -1$

$(\Lambda_p)\times PT$ $\det = +1$ $\Lambda^0_0 \leq -1$

Four disconnected components

$\Lambda^T \eta \Lambda = \eta \rightarrow \eta^{-1}\Lambda^T\eta = \Lambda^{-1}$

$(\Lambda^{-1})^\alpha{}_\beta = (\eta^{-1})^{\alpha\mu}(\Lambda^T)_\mu{}^\nu(\eta)_{\nu\beta} = (\Lambda^T)^\alpha{}_\beta$

Rotation $\Lambda = \begin{pmatrix} 1 & \\ & \begin{matrix} \cos & -\sin \\ \sin & \cos \end{matrix} \end{pmatrix}$ η's $(-)^2$ or $(1)^2$

$\Lambda^T: \theta \to -\theta.$

Boost $\Lambda = \begin{pmatrix} \gamma & \bar{\gamma}v \\ \gamma v & \gamma \end{pmatrix}$ transpose. no effect

Λ's $v \to -v$