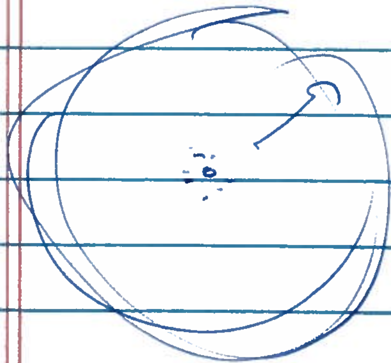
8/29/2016



$$\boxed{|\vec{x}|^2 = t^2}$$

$$x' = \gamma_1(x + v_1 t)$$
$$t' = \gamma_1(t + v_1 x)$$

repeat: $x'' = \gamma_2(x' + v_2 t)$

$t'' = \gamma_2(t' + v_2 x)$.

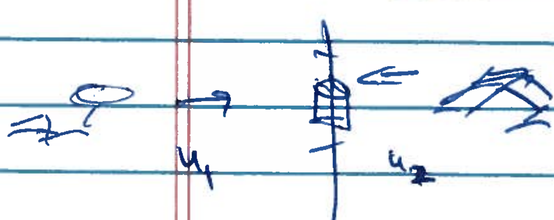
$$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_2 v_2 \\ \gamma_2 v_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1 v_1 \\ \gamma_1 v_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \gamma_1\gamma_2(1+v_1v_2) & \gamma_1\gamma_2(v_1+v_2) \\ \gamma_1\gamma_2(v_1+v_2) & \gamma_1\gamma_2(1+v_1v_2) \end{pmatrix}$$

symmetric in (1, 2).

$$\frac{\gamma v}{\gamma} = \frac{\gamma_1\gamma_2(v_1+v_2)}{\gamma_1\gamma_2(1+v_1v_2)} = \frac{v_1+v_2}{1+v_1v_2}$$

$$\boxed{V = \frac{v_1+v_2}{1+v_1v_2}}$$



$$u_1 \cdot u_2 = -(\gamma_1)(\gamma_2) + (\gamma_1\vec{v}_1)\cdot(\gamma_2\vec{v}_2)$$
$$= -\gamma_1\gamma_2(1 - \vec{v}_1\cdot\vec{v}_2)$$

$= \begin{pmatrix} \gamma_1 \\ \gamma_1 v_1 \end{pmatrix}$ $\quad = \begin{pmatrix} \gamma_2 \\ \gamma_2 \vec{v}_2 \end{pmatrix}$ $\quad = -(\gamma_{21})(1) + (\ )\cdot 0$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} \gamma_{21} \\ \gamma_{21}\vec{v}_{21} \end{pmatrix}$

$$\boxed{\gamma_{21} = \gamma_1\gamma_2(1 - \vec{v}_1\cdot\vec{v}_2)}$$

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$$\Lambda_x = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Lambda_y = \begin{pmatrix} \gamma & 0 & \gamma v & 0 \\ 0 & 1 & 0 & 0 \\ \gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_{(-y)} \Lambda_{(-x)} \Lambda_y \Lambda_x =$$

$$\begin{pmatrix} \gamma^4 - 2\gamma^3 v^2 & \gamma^4 v - \gamma^3 (v + v^2) & \gamma^3 v - \gamma^2 v & 0 \\ -\gamma^3 v + \gamma^2 v & -\gamma^3 v^2 + \gamma^2 & -\gamma^2 v^2 & 0 \\ -\gamma^4 v + \gamma^3 v & -\gamma^4 v^2 + 2\gamma^3 v^2 & -\gamma^3 v^2 + \gamma^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -\frac{1}{2} v^3 & \frac{1}{2} v^3 & 0 \\ -\frac{1}{2} v^3 & 1 & -v^2 & 0 \\ \frac{1}{2} v^3 & v^2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

rotation $\theta = v^2$ about z-axis

boost: $\vec{v} = \frac{1}{2} v^3 (-\hat{x} + \hat{y})$

$$\Lambda = \exp\left(\vec{\theta}\cdot\vec{J} + \vec{\xi}\cdot\vec{K}\right)$$

$$(J_1)^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & & & \\ & 0 & 0 & 0 \\ & 0 & 0 & -1 \\ & 0 & 1 & 0 \end{pmatrix} \quad \text{etc.}$$

$$(K_1)^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & & & \\ 0 & & 0 & \\ 0 & & & \end{pmatrix} \quad \text{etc.}$$

$$[J_i, J_j] = \varepsilon_{ijk} J_k. \quad \text{(rotation)}$$

$$[J_i, K_j] = \varepsilon_{ijk} K_k \quad (\vec{K} \text{ is a "vector" under rotation})$$

$$[K_i, K_j] = -\varepsilon_{ijk} J_k.$$

Pauli. $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0.$$

$\sigma_i \sigma_j = i\sigma_k$ (cyc).

$$\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\,\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = t\cdot\mathbb{1} + \vec{x}\cdot\vec{\sigma} = \underline{x^\mu \sigma_\mu}.$$

$\mathrm{Tr}\, X = 2t$ $\qquad x^\mu = \frac{1}{2}\mathrm{Tr}(\sigma^\mu X)$

$\mathrm{Tr}(X\sigma^i) = \mathrm{Tr}(\sigma^i X) = 2x^i$

real $\vec{x}, t \to$ Hermitian X.

$$\det X = (t+z)(t-z) - (x+iy)(x-iy) = t^2 - \vec{x}^2 = -x\cdot x$$

Let. $\vec{J} = -\frac{i}{2}\vec{\sigma}$

$\vec{K} = -\frac{1}{2}\vec{\sigma}$

$$\underline{[J_i, J_j]} = \left(\frac{-i}{2}\right)^2 [\sigma_i, \sigma_j] = \frac{-1}{4}\cdot 2i\,\varepsilon_{ijk}\sigma_k$$

$$= -\frac{i}{2}\varepsilon_{ijk}\sigma_k = \underline{\varepsilon_{ijk} J_k}$$

$$\underline{[K_i, J_j]} = (-i)[J_i, J_j] = (-i)\varepsilon_{ijk} J_k = \underline{\varepsilon_{ijk} K_k}$$

$$\underline{[K_i, K_j]} = \frac{1}{4}\cdot 2i\,\varepsilon_{ijk}\sigma_k = \frac{i}{2}\varepsilon_{ijk}\sigma_k = \underline{-\varepsilon_{ijk} J_k}$$

$$\Lambda = \exp\left(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K}\right) = \exp\left(-\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\zeta}\cdot\vec{\sigma}\right)$$

$$= \exp\left(-\tfrac{1}{2}(\vec{\zeta}+i\vec{\theta})\cdot\vec{\sigma}\right)$$

let. $\vec{\zeta}+i\vec{\theta} = z\,\hat{n}$

$$z^2 = (\vec{\zeta}+i\vec{\theta})\cdot(\vec{\zeta}+i\vec{\theta}) = \zeta^2-\theta^2+2i\,\vec{\zeta}\cdot\vec{\theta}$$

$$(\hat{n}\cdot\vec{\sigma})^2 = (\hat{n}\cdot\hat{n})\,\mathbb{1} - i\,\cancel{(\hat{n}\times\hat{n})}\cdot\vec{\sigma} = \mathbb{1}.$$

$$(\hat{n}\cdot\vec{\sigma})^{\text{even}} = \mathbb{1} \qquad (\hat{n}\cdot\vec{\sigma})^{\text{odd}} = (\hat{n}\cdot\vec{\sigma})$$

$$\Lambda = \mathbb{1}\left(1+\tfrac{1}{2}\left(\tfrac{z}{2}\right)^2+\tfrac{1}{24}\left(\tfrac{z}{2}\right)^4+\cdots\right)$$

$$-(\hat{n}\cdot\vec{\sigma})\left(\tfrac{z}{2}+\tfrac{1}{6}\left(\tfrac{z}{2}\right)^3+\tfrac{1}{120}\left(\tfrac{z}{2}\right)^5+\cdots\right)$$

$$\boxed{\Lambda = \mathbb{1}\cdot\cosh\tfrac{z}{2} - \hat{n}\cdot\vec{\sigma}\,\sinh\tfrac{z}{2}}$$

$\vec{\zeta}=0$ $\quad \vec{\theta}=\theta\hat{n} \quad z=-i\theta \qquad \Lambda = \mathbb{1}\cdot\cos\tfrac{\theta}{2} - i(\hat{n}\cdot\vec{\sigma})\sin\tfrac{\theta}{2}$

$\vec{\theta}=0$ $\quad \vec{\zeta}=\zeta\hat{n} \quad z=\zeta \qquad \Lambda = \mathbb{1}\cdot\cosh\tfrac{\zeta}{2} - i(\hat{n}\cdot\vec{\sigma})\sinh\tfrac{\zeta}{2}$

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Action of Λ. $\quad \underline{X}' = \Lambda \underline{X} \Lambda^{\dagger}$

$\det \Lambda = 1 \quad \log \det M = \mathrm{Tr} \log M . \quad \mathrm{Tr}\,\vec{\sigma} = 0$

$\boxed{\det \underline{X}' = \det \underline{X}} \quad t'^2 - |\vec{x}'|^2 = t^2 - |\vec{x}|^2$

$\vec{\zeta} = 0 \quad \vec{\phi} = \alpha \hat{z} \qquad \Lambda = \mathbb{1} \cdot \cos\frac{\alpha}{2} - i \sigma_z \sin\frac{\alpha}{2} = \begin{pmatrix} \cos\frac{\alpha}{2} - i \sin\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} + i \sin\frac{\alpha}{2} \end{pmatrix}$

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} t+z & \rho e^{-i\phi} \\ \rho e^{i\phi} & t-z \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

$$= \begin{pmatrix} t+z & \rho e^{-i\phi} e^{-i\alpha} \\ \rho e^{i\phi} e^{i\alpha} & t+z \end{pmatrix}.$$

$\vec{\theta} = 0 \quad \vec{\zeta} = \alpha \hat{z}$

$$\begin{pmatrix} \cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2} & 0 \\ 0 & \cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \begin{pmatrix} - & \cdot \\ & - \end{pmatrix}$$

$$= \begin{pmatrix} (t+z)(\cosh\alpha - \sinh\alpha) & x-iy \\ x+iy & (t-z)(\cosh\alpha + \sinh\alpha) \end{pmatrix}$$

$$t' = \tfrac{1}{2}(t'+z') - \tfrac{1}{2}(t'-z') = \tfrac{1}{2}\big(t\cosh\alpha + \cancel{z\cosh\alpha} - \cancel{t\sinh\alpha} - z\sinh\alpha + t\cosh\alpha - \cancel{z\cosh\alpha} + \cancel{t\sinh\alpha} - z\sinh\alpha \big)$$

$t' = t\cosh\alpha - z\sinh\alpha. \quad = \gamma(t - vz).$