8/31/16

$$\vec{J} = -\frac{i}{2}\vec{\sigma} \qquad \vec{K} = -\frac{1}{2}\sigma.$$

$$\Lambda = \exp(\vec{\theta}\cdot\vec{J} + \vec{\zeta}\cdot\vec{K})$$

$$\underline{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = \underline{x^\mu \sigma_\mu}.$$

$$\underline{X}' = \Lambda \underline{X} \Lambda^\dagger$$

Experiment: $\vec{\zeta} = \alpha\hat{z}$, $\vec{\theta} = \beta\hat{z}$ (parallel).

$$\begin{aligned} t' &= t\cosh\alpha - z\sinh\alpha \\ z' &= z\cosh\alpha - t\sinh\alpha \\ x' &= x\cos\beta - y\sin\beta \\ y' &= y\cos\beta + x\sin\beta \end{aligned}$$

(simultaneously).

$[J_z, K_z] = 0$

Use to show : $B_1 B_2 = R \cdot B'$.

$$\Lambda_1 = \cosh\frac{\zeta_1}{2} - \sinh\frac{\zeta_1}{2}\,\hat{n}_1\cdot\vec{\sigma}$$

$$\Lambda_2 = \cosh\frac{\zeta_2}{2} - \sinh\frac{\zeta_2}{2}\,\hat{n}_2\cdot\vec{\sigma}$$

$$\Lambda = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\,(\hat{n}\cdot\vec{\sigma})$$

$$\Lambda' = \cosh\frac{\zeta'}{2} - \sinh\frac{\zeta'}{2}\,(\hat{n}'\cdot\vec{\sigma})$$

$$(c_1 - s_1\hat{n}_1\cdot\vec{\sigma})(c_2 - s_2\hat{n}_2\cdot\vec{\sigma}) = (c - is\,\hat{n}\cdot\vec{\sigma})(c' - s'\hat{u}'\cdot\vec{\sigma})$$

$$c_1c_2 - c_2s_1\hat{u}_1\cdot\vec{\sigma} - c_1s_2\hat{u}_2\cdot\vec{\sigma}$$

$$+ s_1s_2\,\hat{u}_1\cdot\hat{n}_2 + s_1s_2\,i(\hat{u}_1\times\hat{u}_2)\cdot\vec{\sigma}$$

$$= cc' - cs'\hat{n}'\cdot\vec{\sigma} - ic's\,\hat{n}\cdot\vec{\sigma}$$

$$+ iss'\,\hat{n}\cdot\hat{n}' - ss'(\hat{u}\times\hat{u}')\cdot\vec{\sigma}$$

$\mathbb{1}$:

$$c_1c_2 + s_1s_2(\hat{n}_1\cdot\hat{n}_2) = cc' + i\,ss'\,(\hat{n}\cdot\hat{u}')$$

$\cdot\vec{\sigma}$:

$$-c_2s_1\hat{n}_1 - c_1s_2\hat{n}_2 + is_1s_2\,\hat{n}_1\times\hat{n}_2$$

$$= -cs'\hat{n}' - ic's\hat{n} - ss'\,\hat{n}\times\hat{n}'$$

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* Imaginary part of scalar: $\sqrt{\ } ss'(\hat{n}\cdot\hat{n}') = 0$

In general, $s \neq 0$, $s' \neq 0$ $\Rightarrow$ $\sqrt{\ } \hat{n}\cdot\hat{n}' = 0$ (*)

$s' = 0$ no boost

$s = 0$ no rotation colinear $\hat{n}_1 \times \hat{n}_2 = 0$

* Imaginary part of vector

$s = 0$
$\to \hat{n}_1 \times \hat{n}_2 = 0$

$\sqrt{\ } s_1 s_2 \hat{n}_1 \times \hat{n}_2 = -c' s \hat{n}$ $\sqrt{\ } \hat{n} \parallel \hat{n}_2 \times \hat{n}_1$

magnitude. $\sqrt{\ } c's = s_1 s_2 \sin\theta_{12}$

* Real part of scalar $\sqrt{\ } cc' = c_1 c_2 + s_1 s_2 (\hat{n}_1 \cdot \hat{n}_2)$

divide: $$\tan\frac{\theta}{2} = \frac{\sinh\frac{\zeta_1}{2}\,\sinh\frac{\zeta_2}{2}\,\sin\theta_{12}}{\cosh\frac{\zeta_1}{2}\cosh\frac{\zeta_2}{2} + \sinh\frac{\zeta_1}{2}\sinh\frac{\zeta_2}{2}\cos\theta_{12}}$$

(+) $\cosh(\zeta_1+\zeta_2)$

(−) ?.

denominator ≥ 1. $\theta \neq \pi$ (*)

$$\sqrt{\ } \cosh\frac{\zeta'}{2} = \frac{\sinh\frac{\zeta_1}{2}\,\sinh\frac{\zeta_2}{2}\,\sin\theta_{12}}{\sin\frac{\theta}{2}}$$

* ~~Real part of vector.~~ ~~$cs'\hat{n}' +$~~ ~~$cs'\hat{n} - ic'$~~

Real part of vector

$$c s' \hat{n}' + s s' \hat{n} \times \hat{n}' = c_2 s_1 \hat{n}_1 + c_1 s_2 \hat{n}_2$$

$\hat{n}' \cdot$

$$c s' = c_2 s_1 \hat{n}_1 \cdot \hat{n}' + c_1 s_2 \hat{n}_2 \cdot \hat{n}'$$

$$\hat{n}' = \alpha \hat{n}_1 + \beta \hat{n}_2$$

1-2 plane

$$\alpha c_2 s_1 + \beta c_1 s_2 = c s'$$

$$\hat{n}'^2 = \alpha^2 + \beta^2 + 2\alpha\beta \hat{n}_1 \cdot \hat{n}_2 = 1$$

$\varsigma_1, \varsigma_2 \ll 1$

$$\frac{\theta}{2} \simeq \left(\frac{\varsigma_1}{2}\right)\left(\frac{\varsigma_2}{2}\right)$$

$$\theta \simeq \frac{1}{2} v_1 v_2 \sin\theta_{12}$$

$$= \frac{1}{2} |\vec{v}_1 \times \vec{v}_2|$$

$\varsigma_1, \varsigma_2 \gg 1$

$$\tan\frac{\theta}{2} \simeq \frac{\sin\theta_{12}}{1+\cos\theta_{12}} = \frac{2\sin\frac{\theta_{12}}{2}\cos\frac{\theta_{12}}{2}}{2\cos^2\frac{\theta_{12}}{2}} = \tan\frac{\theta_{12}}{2}$$

$$\theta = \theta_{12}$$

X also works as spinor $\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$t = \frac{1}{2}(|\alpha|^2 + |\beta|^2)$

$z = \frac{1}{2}(|\alpha|^2 - |\beta|^2)$ $\quad \hat{x} = \frac{1}{2}\chi^\dagger \vec{\sigma} \chi$

$x = \frac{1}{2}(\alpha^* \beta^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{2}(\alpha^*\beta + \beta^*\alpha)$

$y = \frac{1}{2}(\alpha^* \beta^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{2}\alpha^*(-i\beta) + \frac{1}{2}\beta^*(i\alpha)$

$$X = \begin{pmatrix} \underbrace{\frac{1}{2}(|\alpha|^2 + \cancel{|\beta|^2}) + \frac{1}{2}(|\alpha|^2 - \cancel{|\beta|^2})}_{|\alpha|^2} & \underbrace{\frac{1}{2}(\cancel{\alpha^*\beta} + \beta^*\alpha) - i\frac{1}{2}(\cancel{-i\alpha^*\beta} + i\beta^*\alpha)}_{\beta^*\alpha} \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

$$= \chi\chi^\dagger = \cancel{(\alpha\ \beta)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}}\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}(\alpha^* \ \beta^*)$$

$\chi' = \Lambda\chi$ $\quad \Lambda = \exp(\vec{\theta}\cdot\vec{J} + \vec{\gamma}\cdot\vec{K})$

$X' = (\Lambda\chi)(\Lambda\chi)^\dagger = \Lambda\chi\chi^\dagger\Lambda^\dagger = \Lambda X \Lambda^\dagger$

$\boxed{\text{Dirac spinor}}$.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\begin{pmatrix} i \\ -i \end{pmatrix}$$

$$\Sigma^{0i} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \Sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$= K^i \qquad J^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$J_1 K_2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} = iK_3$$

(different convention with i's)