9/9/2016 Symmetric, antisymmetric tensors.

$$T_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) + \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) = T_{(\alpha\beta)} + T_{[\alpha\beta]}$$

3d. $3 \times 3 = 9$. $\frac{3 \times 2}{2} = 3$ $\frac{3 \times 2}{2} + 3 = 6$ $\frac{n(n-1)}{2}$ for $\frac{n(n-1)}{2}$ $\frac{n(n+1)}{2}$

4d $4 \times 4 = 16$ 10 $+ 6$.

$\varepsilon_{\alpha\beta} O_{\alpha\beta} = 0$

$$T_{\alpha\beta\gamma} \neq T_{(\alpha\beta\gamma)} + T_{[\alpha\beta\gamma]}$$

4d.

111	4	
112	$4 \cdot 3 = 12$	20
123	4	

$123 = 4$

$4^3 = 64$

$$\eta'_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \eta_{\mu\nu} = \eta_{\alpha\beta}.$$ by construction

$\eta^{\alpha\beta}$ $\delta^\alpha{}_\beta$

$\tilde{\varepsilon}_{0123} = +1$ Carroll (1.68), MTW ...

$$\varepsilon'_{\alpha\beta\gamma\delta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \Lambda_\gamma{}^\rho \Lambda_\delta{}^\sigma \varepsilon_{\mu\nu\rho\sigma}$$

$$= (\det\Lambda) \times (-1)^P = (\det\Lambda) \cdot \varepsilon_{\alpha\beta\gamma\delta}$$

proper. $\varepsilon' = \varepsilon$

Dual (Hodge dual)

$$V_\alpha \rightarrow {}^*V_{\alpha\beta\gamma} = \varepsilon^{\mu}{}_{\alpha\beta\gamma} V_\mu$$

$1 \leftrightarrow 3$

$$V_{\alpha\beta\gamma} \rightarrow {}^*V_\alpha = \frac{1}{6} \varepsilon^{\mu\nu\rho}{}_{\alpha} V_{\mu\nu\rho}$$

$$V_{\alpha\beta} \qquad {}^*V_{\alpha\beta} = \frac{1}{2} \varepsilon^{\mu\nu}{}_{\alpha\beta} V_{\mu\nu}$$

$n \leftrightarrow n-p$

$\frac{1}{p!}$ (symmetries)

$$*(*A) = (-1)^s (-1)^{p(n-p)}$$

$$*(*V_\alpha) = \frac{1}{6} \underbrace{\varepsilon^{\mu\nu\rho}{}_{\alpha} \left(\varepsilon^{\lambda}{}_{\mu\nu\rho}\right.}_{6\delta^\lambda_\alpha} \left. V_\lambda\right)$$

p-forms. (differential forms).

Antisymmetric $(0,p)$ tensor.

$$\alpha_{\mu_1 \cdots \mu_p} = \alpha_{[\mu_1 \cdots \mu_p]}$$

$$\tilde{\alpha} = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} \tilde{\omega}^{\mu_1} \wedge \cdots \wedge \tilde{\omega}^{\mu_p}$$

antisymmetrized outer product.

$$= \alpha_{|\mu_1 \cdots \mu_p|} \tilde{\omega}^{\mu_1} \wedge \cdots \wedge \tilde{\omega}^{\mu_p} \qquad (\underline{\mu_1 < \mu_2 \cdots < \mu_p})$$

$$\underset{(p)}{\tilde{\alpha}} \wedge \underset{(q)}{\tilde{\beta}} = (-1)^{pq} \underset{(q)}{\tilde{\beta}} \wedge \underset{(p)}{\tilde{\alpha}}$$

$\tilde{d}$ "exterior derivative" $\quad p \rightarrow p+1$.

$$\tilde{d} f = f_{,\mu} \tilde{\omega}^{\mu}.$$

$$\tilde{d}(\tilde{\alpha} \wedge \tilde{\beta}) = \underset{p}{(\tilde{d}\tilde{\alpha})} \wedge \underset{q}{\tilde{\beta}} + (-1)^p \tilde{\alpha} \wedge \tilde{d}\tilde{\beta}$$

$$\tilde{d}(x^\alpha) = \frac{\partial x^\alpha}{\partial x^\mu} \tilde{\omega}^\mu = \delta^\alpha{}_\mu \tilde{\omega}^\mu = \tilde{\omega}^\alpha$$

$$\boxed{\tilde{\omega}^\alpha = \tilde{d}(x^\alpha)}$$

$\partial_\mu f = \frac{\partial f}{\partial x^\mu}$

$\boxed{\partial_\mu = \frac{\partial}{\partial x^\mu} = \vec{e}_\mu}$

Carroll. (adapted) (2.73)

$$(\alpha \wedge \beta)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!}\, \alpha_{[\mu_1 \cdots \mu_p}\, \beta_{\mu_{p+1} \cdots \mu_{p+q}]}$$

$$\alpha_\mu, \beta_\nu \longrightarrow (\alpha \wedge \beta)_{\mu\nu} = 2\, \alpha_{[\mu} \beta_{\nu]} = \alpha_\mu \beta_\nu - \alpha_\nu \beta_\mu$$

$$(\mathrm{d}\alpha)_{\mu \mu_1 \cdots \mu_p} = (p+1)\, \partial_{[\mu}\, \alpha_{\mu_1 \cdots \mu_p]}$$

$$\tilde{d}(\tilde{d}\alpha) \longrightarrow \alpha_{[\mu_1 \cdots \mu_p, \underline{\mu\nu}]} = 0 \qquad \tilde{d}^2 = 0.$$

α

$\tilde{F} = \tilde{d}\tilde{A}$

$\tilde{d}\tilde{F} = 0$

$\tilde{d}{}^*\tilde{F} = 4\pi\, {}^*\tilde{J}$

Theorem: $\tilde{d}\tilde{\omega} = 0 \Leftrightarrow \tilde{\omega} = \tilde{d}\tilde{\alpha}$

"exact"

Theorem (Stokes): $\boxed{\int_\Omega d\omega = \int_{\partial\Omega} \omega}$

$$\int_\Omega d^2\omega = \int_{\partial\Omega} d\omega = \int_{\partial^2\Omega} \omega = 0$$

"Boundary of a boundary vanishes"

$\vec{d}\phi \rightarrow$ gradient.

$\vec{d}\phi_\mu \rightarrow \phi_{\mu,\nu} - \phi_{\nu,\mu}$ <u>curl</u>.

$$d\,v_{ij} \rightarrow v_{ij,k} + v_{jk,i} + v_{ki,j}$$
$$- v_{ji,k} - v_{kj,i} - v_{ik,j}$$
$$= 2\left(v_{ij,k} + v_{\cancel{jk},i} + v_{ki,j}\right)$$

$*V_1 = v_{23}$ etc.

$$\rightarrow (*v_k)_{,k} + (*v_i)_{,i} + (*v_j)_{,j}$$

$$\rightarrow \vec{\nabla}\cdot\vec{V}$$

$$\int_a^b dx \frac{\partial F}{\partial x} = F(b) - F(a).$$

$$\cancel{\oint d^2a\,\hat{n}\cdot(\vec{\nabla}\times\vec{v})} \quad \oint_S \vec{v}\cdot d\vec{l} = \int_S d^2a\,\hat{n}\cdot(\vec{\nabla}\times\vec{v})$$

$$\oint_S da\,\hat{n}\cdot\vec{v} = \int_V d^3x\,\vec{\nabla}\cdot\vec{v}$$