9/19/2016

$S = \int d^4x \, \mathcal{L}(\phi_k, \partial_\mu \phi_k)$ (~~many~~ multiple scalar fields)

$\frac{d\phi_k}{d\alpha} \neq 0$ $\frac{d\mathcal{L}}{d\alpha} = 0$ $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$

$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \frac{d\phi_k}{d\alpha}$ $(\sum_k)$

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \frac{d\phi_k}{d\alpha} \right)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \cdot \frac{d\phi_k}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \left(\frac{d\phi_k}{d\alpha} \right)$$

$$= \frac{\partial \mathcal{L}_k}{\partial \phi} \frac{d\phi_k}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{d}{d\alpha}(\partial_\mu \phi_k)$$

$$= \frac{d\mathcal{L}}{d\alpha} = 0.$$

Symmetry. $\frac{d}{d\alpha} = 0$ $\Rightarrow$ Conservation $\partial_\mu J^\mu = 0$

modern thinking. all conservation.

EM. $\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}$. $\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{4\pi} F^{\alpha\beta}}$

$$A'_\beta = \Lambda_\beta{}^\lambda A_\lambda = \exp\left(\alpha^{\mu\nu} (M_{\mu\nu})_\beta{}^\lambda \right) A_\lambda$$

$$\frac{\partial A_\beta}{\partial \alpha^{\mu\nu}} = (M_{\mu\nu})_\beta{}^\lambda A_\lambda = -\eta_{\mu\beta} A_\nu + \eta_{\nu\beta} A_\mu$$

$$\hookrightarrow -\eta_{\mu\beta}\eta^\lambda{}_\nu + \eta_{\nu\beta}\eta^\lambda{}_\mu$$

$$S_{\mu\nu}{}^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} \frac{\partial A_\beta}{\partial \alpha^{\mu\nu}} = -\frac{1}{4\pi} F^{\alpha\beta} \left(-\eta_{\mu\beta} A_\nu + \eta_{\nu\beta} A_\mu \right)$$

$$\boxed{S_{\mu\nu}{}^\alpha = F^\alpha{}_\mu A_\nu - F^\alpha{}_\nu A_\mu}$$

Spin density

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$J_{\mu\nu}{}^\alpha = x_\mu T_\nu{}^\alpha - x_\nu T_\mu{}^\alpha + S_{\mu\nu}{}^\alpha \quad \text{(orbital + spin)}$$

$$\partial_\alpha J_{\mu\nu}{}^\alpha = \eta_{\mu\alpha} T_\nu{}^\alpha + \cancel{x_\mu \partial_\alpha T_\nu{}^\alpha} - \eta_{\nu\alpha} T_\mu{}^\alpha - \cancel{x_\mu T_\nu{}^\alpha{}_{,\alpha}}$$

$$= \cancel{T_{\mu\nu}} \; T_{\nu\mu} - T_{\mu\nu} + \partial_\alpha S_{\mu\nu}{}^\alpha = 0$$

$$\boxed{T_{\mu\nu} - T_{\nu\mu} = \partial_\alpha S_{\mu\nu}{}^\alpha}$$

$$G_{\mu\nu\rho} = \frac{1}{2}\left(S_{\nu\rho\mu} + S_{\mu\rho\nu} - S_{\mu\nu\rho} \right)$$

$$G_{\mu\nu\rho} + G_{\mu\rho\nu} = \frac{1}{2}\Big(\cancel{S_{\nu\rho\mu}} + S_{\mu\rho\nu} - \cancel{S_{\mu\nu\rho}} + \cancel{S_{\rho\nu\mu}} + S_{\mu\nu\rho} - S_{\mu\rho\nu} \Big) = 0$$

$S_{\mu\nu\rho} = S_{[\mu\nu]\rho}$ $\qquad$ $G_{\mu\nu\rho} = G_{\mu[\nu\rho]}$

Let $\boxed{\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho G^{\mu\nu\rho}}$

$$\boxed{\partial_\nu \Theta^{\mu\nu} = \partial_\nu T^{\mu\nu} + \underbrace{\partial_\nu \partial_\rho G^{\mu\nu\rho}}_{=0} = 0}$$

conserved current.

$$\int d^3x\, \Theta^{\mu 0} = \int d^3x\, T^{\mu 0} + \int d^3x\, \partial_\rho G^{\mu 0 \rho}$$

$$= \int d^3x\, T^{\mu 0} + \int d^3x\, \partial_k G^{\mu 0 k}$$

$$\longrightarrow \oint d\vec{a}\, \hat{n}_k\, G^{\mu 0 k} \to 0.$$

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} + \partial_\rho \left(G^{\mu\nu\rho} - G^{\nu\mu\rho} \right)$$

$$\frac{1}{2}\Big(\cancel{S^{\nu\rho\mu}} + S^{\mu\rho\nu} - S^{\mu\nu\rho} - S^{\mu\rho\nu} - \cancel{S^{\nu\rho\mu}} + S^{\nu\mu\rho} \Big) = -S^{\mu\nu\rho}$$

$$= T^{\mu\nu} - T^{\nu\mu} - \partial_\rho S^{\mu\nu\rho} = \partial_\rho J^{\mu\nu\rho} \to 0$$

Symmetric

Canonical

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}(\partial^\mu A_\lambda) + \eta^{\mu\nu}\mathcal{L}$$

$$= \frac{1}{4\pi} F^{\nu\lambda}\,\partial^\mu A_\lambda - \frac{1}{16\pi}\eta^{\mu\nu} F_{\lambda\kappa}F^{\lambda\kappa}$$

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho G^{\mu\nu\rho}$$

$$G_{\mu\nu\rho} = \frac{1}{2}(S_{\nu\rho\mu} + S_{\mu\rho\nu} - S_{\mu\nu\rho})$$

$$S_{\mu\nu\rho} = F_{\rho\mu}A_\nu - F_{\rho\nu}A_\mu$$

Symmetrize

GR: $\Theta^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$

$$\Theta^{\mu\nu} = \frac{1}{4\pi} F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{16\pi} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} = T^{\mu\nu}$$

$$T^{00} = \frac{1}{8\pi}(E^2 + B^2) \qquad \frac{1}{2}(E^2+B^2) \qquad \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right)$$

$$T^{0i} = T^{i0} = \frac{1}{4\pi}(\vec{E} \times \vec{B})^i$$

$$T^{ij} = -\frac{1}{4\pi}\left(E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(E^2+B^2)\right)$$

$$\vec{E} = E\hat{z} \rightarrow T^{ij} = -\frac{E^2}{4\pi}\begin{pmatrix} -\frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{E^2}{8\pi} & & \\ & \frac{E^2}{8\pi} & \\ & & -\frac{E^2}{8\pi} \end{pmatrix}$$

} pressure $\perp$ field lines

$\leftarrow$ tension along field lines

MHD. $\frac{B^2}{2\mu} \leftrightarrow$ magnetic pressure

$\frac{B^2}{8\pi}$

$$T^{\mu}{}_{\mu} = -\frac{(E^2+B^2)}{8\pi} + \frac{(E^2+B^2)}{8\pi}(3) - \frac{1}{4\pi}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})$$

δ^{ii}

$= 0$