9/21/2016.

Carroll. "similarly, from (1.162) . . .

$$\boxed{T^{\mu\nu}_{EM} = \frac{1}{4\pi} F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{16\pi} \eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}}$$

$$\delta^{\mu}_{\mu} = 4$$

$$T^{\mu}_{\mu} = \frac{1}{4\pi} F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{16\pi} \eta^{\mu}_{\mu} F^{\lambda\sigma} F_{\lambda\sigma} = 0.$$

conformal gravity.

General Coordinates. (Ch. 2)

repeat

$x^{\mu}(\lambda)$ 1-d. curve. $f(\lambda) = f(x^{\mu}(\lambda))$ = scalar.

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^{\mu}} \frac{dx^{\mu}}{d\lambda}$$ $\quad x^{\mu} = x^{\mu}(x'^{\alpha})$ Think: r, θ, ϕ.

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^{\mu}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{dx'^{\beta}}{d\lambda}$$ inner x'

chain rule.

$$\left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\right)\left(\frac{\partial x^{\mu}}{\partial x'^{\beta}}\right) = \frac{\partial x'^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta}.$$ matrix inverses.

$\hat{V} = V^{\mu} \hat{e}_{\mu}$. vector. $\quad \hat{u} = \frac{dx^{\mu}}{d\lambda} \hat{e}_{\mu}$

$\tilde{w} = w_{\nu} \tilde{\omega}^{\nu}$ 1-form $\quad \tilde{df} = \frac{\partial f}{\partial x^{\mu}} \tilde{\omega}^{\mu}$

$$\langle \tilde{d}f, \hat{u} \rangle = \left\langle \frac{\partial f}{\partial x^\mu}\tilde{\omega}^\mu, \frac{dx^\nu}{d\lambda}\hat{e}_\nu \right\rangle$$

$$= \frac{\partial f}{\partial x^\mu}\frac{dx^\nu}{d\lambda}\underbrace{\langle \tilde{\omega}^\mu, \hat{e}_\nu \rangle}_{\delta^\mu_\nu} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \nabla_u f = \frac{df}{d\lambda}$$

$$\langle \tilde{d}f, \hat{u} \rangle = \left\langle \frac{\partial f}{\partial x'^\alpha}\underbrace{\frac{\partial x'^\alpha}{\partial x^\mu}\tilde{\omega}^\mu}, \frac{\partial x^\nu}{\partial x'^\beta}\frac{dx'^\beta}{d\lambda}\underbrace{\hat{e}_\nu} \right\rangle$$

$$= \left\langle \frac{\partial f}{\partial x'^\alpha}\tilde{\omega}'^\alpha, \frac{dx'^\beta}{d\lambda}\hat{e}'_\beta \right\rangle \qquad \langle \tilde{\omega}'^\alpha, \hat{e}'_\beta \rangle = \delta^\alpha_\beta$$

$$\boxed{\tilde{\omega}'^\alpha = \frac{\partial x'^\alpha}{\partial x^\mu}\tilde{\omega}^\mu} \qquad \boxed{\hat{e}'_\beta = \frac{\partial x^\nu}{\partial x'^\beta}\hat{e}_\nu}$$

$f = x^\alpha$ $\quad \tilde{d}f = \frac{\partial f}{\partial x^\beta}\tilde{\omega}^\beta = \delta^\alpha_\beta\tilde{\omega}^\beta$ $\quad \boxed{\tilde{\omega}^\alpha = \tilde{d}x^\alpha}$

$\nabla_u f = u^\alpha \frac{\partial f}{\partial x^\alpha} = \left(u^\alpha \frac{\partial}{\partial x^\alpha}\right) f$ $\qquad \boxed{\hat{e}_\beta : \frac{\partial}{\partial x^\beta}}$ $\qquad$ ($\hat{e}$ transforms like ∂_α)

All defined w/o "metric".

metric maps 2 vectors $\to$ scalar. $\quad \hat{e}_\alpha \cdot \hat{e}_\beta = g_{\alpha\beta}$ $\quad (0,2)$

$$\hat{A}\cdot\hat{B} = (A^\mu \hat{e}_\mu)\cdot(B^\nu \hat{e}_\nu) = A^\mu B^\nu (\hat{e}_\mu \cdot \hat{e}_\nu)$$

$$= A^\mu B^\nu g_{\mu\nu} \qquad \langle \hat{e}_\alpha \cdot \tilde{g} \cdot \hat{e}_\beta \rangle = g_{\alpha\beta}$$

$$\tilde{g}(\hat{e}_\alpha, \hat{e}_\beta)$$

$$\hat{A}\cdot\hat{B} = \underbrace{A^{\mu}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}}\;\underbrace{B^{\nu}\frac{\partial x'^{\beta}}{\partial x^{\nu}}}\;\underbrace{\left(\frac{\partial x^{\rho}}{\partial x'^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\beta}}\,g_{\rho\sigma}\right)}_{g'_{\alpha\beta}}$$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}}$$

$$\frac{\partial x^{i}}{\partial x'^{j}} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}$$

$$\frac{\partial x'^{i}}{\partial x^{j}} = \frac{1}{\underset{r}{\det}}\cdot\begin{pmatrix} r\cos\phi & +r\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix}$$

$$\tilde{d}x = \frac{\partial x}{\partial r}\tilde{d}r + \frac{\partial x}{\partial\phi}\tilde{d}\phi = \cos\phi\,\tilde{\omega}^{r} - r\sin\phi\,\tilde{\omega}^{\phi}$$

$$\tilde{d}y = \frac{\partial y}{\partial r}\tilde{d}r + \frac{\partial y}{\partial\phi}\tilde{d}\phi = \sin\phi\,\tilde{\omega}^{r} + r\cos\phi\,\tilde{\omega}^{\phi}$$

$$\tilde{\tilde{g}} = \delta_{ij}\,\tilde{d}x^{i}\otimes\tilde{d}x^{j} = \tilde{d}x\otimes\tilde{d}x + \tilde{d}y\otimes\tilde{d}y$$

$$= (\cos\phi\,\tilde{\omega}^{r} - r\sin\phi\,\tilde{\omega}^{\phi})\otimes(\cos\phi\,\tilde{\omega}^{r} - r\sin\phi\,\tilde{\omega}^{\phi})$$

$$+ (\sin\phi\,\tilde{\omega}^{r} + r\cos\phi\,\tilde{\omega}^{\phi})\otimes(\sin\phi\,\tilde{\omega}^{r} + r\cos\phi\,\tilde{\omega}^{\phi})$$

$$= (\cos^2\phi + \sin^2\phi)\,\tilde{\omega}^{r}\otimes\tilde{\omega}^{r} + \cancel{(\cos\phi(-r\sin\phi) + \sin\phi(r\cos\phi))}\,\tilde{\omega}^{r}\otimes\tilde{\omega}^{\phi}$$

$$+ \cancel{(-r\sin\phi\cdot\cos\phi + r\cos\phi\cdot\sin\phi)}\,\tilde{\omega}^{\phi}\otimes\tilde{\omega}^{r} + (r^2\sin^2\phi + r^2\cos^2\phi)\,\tilde{\omega}^{\phi}\otimes\tilde{\omega}^{\phi}$$

$$= \tilde{\omega}^{r}\otimes\tilde{\omega}^{r} + r^2\,\tilde{\omega}^{\phi}\otimes\tilde{\omega}^{\phi} = \underline{\tilde{d}r\otimes\tilde{d}r + r^2\,\tilde{d}\phi\otimes\tilde{d}\phi}$$

4

$$g'_{ij} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}\delta_{mn} = \left(\frac{\partial x}{\partial x'}\right)^T \cdot \mathbb{I}\left(\frac{\partial x}{\partial x'}\right)$$

$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cancel{-\cos\phi\, r\sin\phi + \sin\phi\, r\cos\phi} \\ \cancel{-r\sin\phi\cos\phi + r\cos\phi\sin\phi} & r^2\sin^2\phi + r^2\cos^2\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

unit

$$\tilde{\tilde{g}} = \tilde{d}x \otimes \tilde{d}x + \tilde{d}y \otimes \tilde{d}y = \tilde{d}r \otimes \tilde{d}r + r^2\,\tilde{d}\phi \otimes \tilde{d}\phi$$

$$\Leftrightarrow \quad \left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$\Leftrightarrow \quad \boxed{ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2}$$

Two representations.

$$\tilde{\tilde{g}} = 1\cdot(\tilde{d}r \otimes \tilde{d}r) + r^2(\tilde{d}\phi \otimes \tilde{d}\phi)$$

coordinate basis $\tilde{\omega}^\alpha = \tilde{d}x^\alpha$

$$\tilde{\tilde{g}} = (\tilde{d}r) \otimes (\tilde{d}r) + (r\tilde{d}\phi) \otimes (r\tilde{d}\phi)$$

"local orthonormal ~~frame~~ basis"

$g_{\alpha\beta} = \tilde{g}(\hat{e}_\alpha, \hat{e}_\beta) = \langle \hat{e}_\alpha \, \tilde{g} \, \hat{e}_\beta \rangle = \hat{e}_\alpha \cdot \tilde{g} \cdot \hat{e}_\beta .$

g maps vector → form

$\hat{A} = A^\mu \hat{e}_\mu \qquad \hat{A} \cdot \tilde{g} = (A^\mu \hat{e}_\mu) \cdot (g_{\alpha\beta} \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta)$

$= g_{\alpha\beta} A^\mu (\delta^\alpha_\mu) \tilde{\omega}^\beta = (g_{\mu\beta} A^\mu) \tilde{\omega}^\beta$

$\tilde{A} = (g_{\mu\beta} A^\mu) \tilde{\omega}^\beta = \underline{A_\beta \tilde{\omega}^\beta}$

$A_\alpha = g_{\alpha\beta} A^\beta \qquad \underline{g_{\alpha\beta} = g_{\beta\alpha}}$

nondegenerate (invertible) → $\det \frac{\partial x^{\alpha'}}{\partial x^\beta} \neq 0 .$

$\Rightarrow \boxed{g = \det g_{\mu\nu} \neq 0}$

$\to \boxed{A^\alpha = (g^{-1})^{\alpha\beta} A_\beta = g^{\alpha\beta} A_\beta}$