9/23/2016

$u^\mu = \frac{dx^\mu}{d\lambda} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{dx'^\alpha}{d\lambda}$

$\partial_\mu f = \frac{\partial f}{\partial x^\mu} = \frac{\partial f}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\mu}$

In general, don't preserve η.

$\hat{V} = V^\alpha \hat{e}_\alpha$ rs, sinks.

$\longleftrightarrow$ $$\boxed{g_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\eta_{\mu\nu}}$$

$\hat{e}_\alpha = \frac{\partial x^\mu}{\partial x'^\alpha}\hat{e}_\mu$

$$\tilde{\tilde{g}} = g_{\alpha\beta}\, \tilde{d}x^\alpha \otimes \tilde{d}x^\beta$$

$$\Longleftrightarrow ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$x = r\cos\phi$
$y = r\sin\phi$ $\longrightarrow$ $g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$$\underline{\hat{V}\cdot\hat{V} = g_{\alpha\beta} V^\alpha V^\beta}$$

coordinate basis.
V^r V^ϕ not even same units.

$\partial_i f = \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \phi} \end{pmatrix}$ not same units.

$\partial^i f = g^{ij}\partial_j f \rightarrow \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r^2}\frac{\partial f}{\partial \phi} \end{pmatrix}$ $= \frac{\partial f}{\partial \phi}$ $(g^{\phi\phi})^{1/2}\partial_\phi f$

$\partial_{\hat{r}} f = \frac{\partial f}{\partial r}$ $\quad \partial_{\hat{\phi}} f = (\partial^\phi f \cdot \partial_\phi f)^{1/2} = (\cancel{g^{\phi\phi}}$

New trouble. derivatives.

$$\hat{V} = V^\alpha \hat{e}_\alpha . \qquad \frac{\partial \hat{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \hat{e}_\alpha + V^\alpha \cdot \frac{\partial \hat{e}_\alpha}{\partial x^\beta}$$

$$\frac{\partial \hat{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta} \hat{e}_\mu$$ (vector-like object → linear combination of $\hat{e}$'s)

$$\frac{\partial \hat{V}}{\partial x^\beta} = \left(V^\mu{}_{,\beta} + \Gamma^\mu_{\alpha\beta} V^\alpha \right) \hat{e}_\mu$$

$$\boxed{\nabla_\beta V^\mu = V^\mu{}_{;\beta} \equiv V^\mu{}_{,\beta} + \Gamma^\mu_{\alpha\beta} V^\alpha}$$

"Covariant derivative".

$V^\mu{}_{,\alpha}$ ≠ tensor. $\Gamma^\mu_{\alpha\beta}$ ≠ tensor $V^\mu{}_{;\alpha}$ is tensor

$$\nabla \hat{V} = (1,1) \text{ tensor } \Rightarrow (\nabla \hat{V})^\alpha{}_\beta = V^\alpha{}_{;\beta}$$

$x = r\cos\phi$
$y = r\sin\phi$

$$\hat{e}_r = \frac{\partial x}{\partial r} \hat{e}_x + \frac{\partial y}{\partial r} \hat{e}_y . = \cos\phi\, \hat{e}_x + \sin\phi\, \hat{e}_y .$$

$$\hat{e}_\phi = \frac{\partial x}{\partial \phi} \hat{e}_x + \frac{\partial y}{\partial \phi} \hat{e}_y = -r\sin\phi\, \hat{e}_x + r\cos\phi\, \hat{e}_y .$$

$$\frac{\partial \hat{e}_r}{\partial r} = 0 \qquad \frac{\partial \hat{e}_r}{\partial \phi} = -\sin\phi\, \hat{e}_x + \cos\phi\, \hat{e}_y = \frac{1}{r}\hat{e}_\phi$$

$$\frac{\partial \hat{e}_\phi}{\partial r} = \frac{1}{r}\hat{e}_\phi \qquad \frac{\partial \hat{e}_\phi}{\partial \phi} = -r\cos\phi\, \hat{e}_x - r\sin\phi\, \hat{e}_y = -r\,\hat{e}_r$$

$$\frac{\partial \hat{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\,\hat{e}_\mu \qquad \frac{\partial \hat{e}_i}{\partial x^j} = \Gamma^k_{ij}\,\hat{e}_k. \qquad \boxed{\Gamma^r_{rr} = \Gamma^\phi_{rr} = 0}$$

$i=j=r$ $\qquad \frac{\partial \hat{e}_r}{\partial r}$

$i=r$ $\; j=\phi$ $\qquad \boxed{\Gamma^r_{r\phi} = 0} \qquad \boxed{\Gamma^\phi_{r\phi} = \frac{1}{r}}$

$\frac{\partial \hat{e}_r}{\partial \phi}.$

$\frac{\partial \hat{e}_\phi}{\partial r}$ $\qquad \boxed{\Gamma^r_{\phi r} = 0 \qquad \Gamma^\phi_{\phi r} = \frac{1}{r}} \qquad \boxed{\Gamma^r_{\phi\phi} = -r} \qquad \boxed{\Gamma^\phi_{\phi\phi} = 0}$

$\frac{\partial \hat{e}_\phi}{\partial \phi}$

$$\nabla\Phi = \nabla_\mu \Phi \qquad \nabla\cdot\nabla\Phi = \nabla^2\Phi = (\Phi^{;\mu})_{;\mu}$$

$$\Phi^r_{\;;r} = \Phi^r_{\;,r} + \cancel{\Gamma^r_{rr}\Phi^r} + \cancel{\Gamma^\phi_{\phi r}}\,\Phi^{,\phi}$$

$$\Phi^\phi_{\;;\phi} = \Phi^\phi_{\;,\phi} + \Gamma^\phi_{r\phi}\Phi^{,r} + \cancel{\Gamma^\phi_{\phi\phi}}\,\Phi^{,\phi}$$

$$\nabla^2\Phi = (1)\left(\frac{\partial\Phi}{\partial r}\right)_{,r} + \left(\frac{1}{r^2}\frac{\partial\Phi}{\partial\phi}\right)_{,\phi} + \left(\frac{1}{r}\right)(1)\left(\frac{\partial\Phi}{\partial r}\right)$$

$$= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2\Phi}{\partial\phi^2}.$$

∇ acts on $\tilde{\omega}$'s ,

$$\langle \tilde{\omega}^{\mu}, \hat{e}_{\nu} \rangle = \delta^{\mu}{}_{\nu} \qquad \Gamma^{\alpha}{}_{\nu\beta}\, \hat{e}_{\alpha}$$

$$\frac{\partial}{\partial x^{\beta}} \langle \tilde{\omega}^{\mu}, \hat{e}_{\nu} \rangle = \left\langle \frac{\partial \tilde{\omega}^{\mu}}{\partial x^{\beta}}, \hat{e}_{\nu} \right\rangle + \left\langle \tilde{\omega}^{\mu}, \frac{\partial \hat{e}_{\nu}}{\partial x_{\beta}} \right\rangle$$

$$\Gamma'^{\mu}{}_{\alpha\beta}\, \tilde{\omega}^{\alpha}$$

$$\Gamma'^{\mu}{}_{\nu\beta}\, \delta^{\alpha}{}_{\nu} + \Gamma^{\alpha}{}_{\nu\beta}\, \delta^{\mu}{}_{\alpha} = 0$$

$$\boxed{\Gamma'^{\mu}{}_{\nu\beta} = -\Gamma^{\mu}{}_{\nu\beta}}$$

$$\boxed{\frac{\partial \tilde{\omega}^{\mu}}{\partial x^{\beta}} = -\Gamma^{\mu}{}_{\nu\beta}\, \tilde{\omega}^{\nu}}$$

~~$\tilde{w} = w_{\mu} \tilde{\omega}^{\mu}$~~

$$\frac{\partial \tilde{w}}{\partial x^{\beta}} = \frac{\partial}{\partial x^{\beta}} \left(w_{\mu} \tilde{\omega}^{\mu} \right) = \left(\frac{\partial w_{\mu}}{\partial x^{\beta}} \cdot \tilde{\omega}^{\mu} + w_{\mu} \cdot \frac{\partial \tilde{\omega}^{\mu}}{\partial x^{\beta}} \right)$$

$$= \left(\frac{\partial w_{\nu}}{\partial x^{\beta}} - \Gamma^{\mu}{}_{\nu\beta}\, w^{\nu} \right) \tilde{\omega}^{\nu}$$

$$\boxed{w_{\mu;\beta} = (\nabla \tilde{w})_{\mu\beta} = w_{\mu,\beta} - \Gamma^{\mu}{}_{\nu\beta}\, w^{\nu}}$$