9/26/2016. Covariant derivative

$$\partial_\mu V^\nu \to \partial_{\mu'} V^{\nu'} = \left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\right)\left(\frac{\partial x'^\nu}{\partial x^\beta} V^\beta\right)$$

$$= \frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial x'^\nu}{\partial x^\beta}\right)\frac{\partial V^\beta}{\partial x^\alpha} + V^\beta \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta}$$

$$\boxed{\nabla_\mu V^\nu = V^\nu{}_{;\mu} = \partial_\mu V^\nu + \Gamma^\nu_{\lambda\mu} V^\lambda}$$ (3.2) (3.5)

MTW. (8.21) $V^\beta{}_{;\gamma} = V^\beta{}_{,\gamma} + \Gamma^\beta{}_{\mu\gamma} V^\mu$.

Schutz. (5.49) $V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\mu\beta} V^\mu$

Wald (3.1.15) $\nabla_a t^b = \partial_a t^b + \Gamma^b{}_{ac} V^c$

LPPT. (4.7) $Q^\alpha{}_{;\sigma} = Q^\alpha{}_{,\sigma} + \Gamma^\alpha{}_{\nu\sigma} Q^\nu$

Weinberg (4.6.4) $V^\mu{}_{;\lambda} = V^\mu{}_{,\lambda} + \Gamma^\mu{}_{\lambda\kappa} V^\kappa$

linear. $\nabla(\alpha A + \beta B) = \alpha \nabla A + \beta \nabla B$.

Leibnitz. $\nabla(AB) = (\nabla A)B + A(\nabla B)$

2

$$\nabla_\sigma T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} = \partial_\sigma T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}$$

$$+ \Gamma^{\mu_1}{}_{\lambda\sigma} T^{\lambda\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + \Gamma^{\mu_2}{}_{\lambda\sigma} T^{\mu_1\lambda\cdots}{}_{\nu_1\nu_2\cdots}$$

$$- \Gamma^{\lambda}{}_{\nu_1\sigma} T^{\mu_1\mu_2\cdots}{}_{\lambda\nu_2\cdots} - \Gamma^{\lambda}{}_{\nu_2\sigma} T^{\mu_1\mu_2\cdots}{}_{\nu_1\lambda\cdots}$$

$$\boxed{\frac{\partial \hat{e}_\alpha}{\partial x^\beta} = \Gamma^{\mu}{}_{\alpha\beta}\, \hat{e}_\mu}$$ Don't need to know

Cartesian. $\underline{\Gamma = 0}$. $u \cdot v = \eta_{\alpha\beta} u^\alpha v^\beta$

$$\nabla_\mu (u \cdot v) = (u^\alpha{}_{;\mu}) v_\alpha + u_\beta (v^\beta{}_{,\mu})$$

can retain. $\nabla_\mu (u \cdot v) = u^\alpha{}_{;\mu} v_\alpha + u_\beta v^\beta{}_{;\mu}$

$$\boxed{g_{\alpha\beta;\mu} = 0}$$ "<u>metric compatible</u>"

$$\nabla_\rho g_{\mu\nu} = g_{\mu\nu,\rho} - \Gamma^\lambda{}_{\mu\rho} g_{\lambda\nu} - \Gamma^\lambda{}_{\nu\rho} g_{\mu\lambda}$$

$$\nabla_\mu g_{\nu\rho} = g_{\nu\rho,\mu} - \Gamma^\lambda{}_{\nu\mu} g_{\lambda\rho} - \Gamma^\lambda{}_{\rho\mu} g_{\nu\lambda}$$

$$\nabla_\nu g_{\rho\mu} = g_{\rho\mu,\nu} - \Gamma^\lambda{}_{\rho\nu} g_{\lambda\mu} - \Gamma^\lambda{}_{\mu\nu} g_{\rho\lambda}$$

③

$$-\Gamma^{\lambda}{}_{\mu\rho}\, g_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\rho}\, g_{\mu\lambda} = -g_{\mu\nu,\rho}$$

$$+\Gamma^{\lambda}{}_{\nu\mu}\, g_{\lambda\rho} + \Gamma^{\lambda}{}_{\rho\mu}\, g_{\nu\lambda} = g_{\nu\rho,\mu}$$

$$+\Gamma^{\lambda}{}_{\rho\nu}\, g_{\lambda\mu} + \Gamma^{\lambda}{}_{\mu\nu}\, g_{\rho\lambda} = g_{\rho\mu,\nu}$$

$$g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}$$

$$= \Gamma_{\rho\nu\mu} + \Gamma_{\nu\rho\mu} + \Gamma_{\mu\rho\nu} + \Gamma_{\rho\mu\nu}$$

$$- \Gamma_{\nu\mu\rho} - \Gamma_{\mu\nu\rho}$$

$$= 2\,\Gamma_{\rho(\mu\nu)} + \cancel{2\,\Gamma_{\nu[\rho\mu]}} + \cancel{2\,\Gamma_{\mu[\rho\nu]}}$$

"Torsion free": $\Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha(\beta\gamma)}$

$$\boxed{\Gamma^{\sigma}{}_{\mu\nu} = \tfrac{1}{2}\, g^{\sigma\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right)}$$

(3.27)

* One of the most important formulas of 660. "Coordinate frame"

(4)

Antisymmetric part :

let $\hat{u} = u^\mu \hat{e}_\mu$ $\hat{v} = v^\nu \hat{e}_\mu$

$\boxed{\partial_\alpha \hat{e}_\beta = \Gamma^\mu_{\alpha\beta} \hat{e}_\mu}$

$\boxed{\nabla_\alpha \hat{e}_\beta \neq 0}$

$[\hat{u}, \hat{v}] = \nabla_u \hat{v} - \nabla_v \hat{u}$

$= \left(u^\rho v^\mu{}_{;\rho} - v^\rho u^\mu{}_{;\rho} \right) \hat{e}_\mu$

$= \left[u^\rho \left(v^\mu{}_{,\rho} + \Gamma^\mu_{\lambda\rho} v^\lambda \right) - v^\rho \left(u^\mu{}_{,\rho} + \Gamma^\mu_{\lambda\rho} u^\lambda \right) \right] \hat{e}_\mu$

Do for $[\hat{e}_\alpha, \hat{e}_\beta]$ $(\hat{e}_\alpha)^\mu = \delta^\mu_\alpha$, $\frac{\partial}{\partial x^\rho} \to 0$

$[\hat{e}_\alpha, \hat{e}_\beta] = \left[\delta^\rho_\alpha \cdot \Gamma^\mu_{\lambda\rho} \delta^\lambda_\beta - \delta^\rho_\beta \cdot \Gamma^\mu_{\lambda\rho} \delta^\lambda_\alpha \right) \hat{e}_\mu$

$= \left(\Gamma^\mu_{\beta\alpha} - \Gamma^\mu_{\alpha\beta} \right) \hat{e}_\mu$

$$\boxed{[\hat{e}_\alpha, \hat{e}_\beta] = -2 \Gamma^\mu_{[\alpha\beta]} \hat{e}_\mu}$$

(8)

"Holonomic" $c^{\mu}{}_{\alpha\beta} = 0$. (coordinate)

Non-holonomic. Example. $\tilde{\omega}^{\hat{r}} = \tilde{d}r$ $\tilde{\omega}^{\hat{\phi}} = r\,\tilde{d}\phi$

$$\tilde{g} = \tilde{\omega}^{\hat{r}} \otimes \tilde{\omega}^{\hat{r}} + \tilde{\omega}^{\hat{\phi}} \otimes \tilde{\omega}^{\hat{\phi}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c_{\hat{r}\hat{\phi}}{}^{\hat{\phi}} = -c_{\hat{\phi}\hat{r}}{}^{\hat{\phi}} = -\frac{1}{r}$$

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)$$

$$\qquad + \frac{1}{2}\left(c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}\right)$$

$\sigma(\beta\gamma)$ $\quad$ $\uparrow [\alpha\beta]$

Never. Both.

$c_{\alpha\beta\gamma}$. only to work in orthonormal frame.