9/28/2016

$$\boxed{V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta} V^{\mu}} \quad (3.5)$$

$$\boxed{\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} \left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma} \right)} \quad (3.27)$$

$\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{(\alpha\beta)}$

$4 \times 10 = 40$ (4d)

$\boxed{2 \times 3 = 6}$ (2d).

$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$\cancel{\Gamma^{r}{}_{rr}}$, $\cancel{\Gamma^{r}{}_{r\phi}} = \cancel{\Gamma^{r}{}_{\phi r}}$, $\Gamma^{r}{}_{\phi\phi}$

$\cancel{\Gamma^{\phi}{}_{rr}}$ $\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r}$ $\cancel{\Gamma^{\phi}{}_{\phi\phi}}$.

only $g_{\phi\phi}$ has derivatives $\rightarrow$ 2 ϕ's.

$g_{\phi\phi}$ has only r-derivative $\rightarrow$ 1 r

$$\Gamma^{r}{}_{\phi\phi} = \frac{1}{2} g^{rr} \left(\cancel{g_{r\phi,\phi}} + \cancel{g_{r\phi,\phi}} - g_{\phi\phi,r} \right) = -r$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{2} g^{\phi\phi} \left(\cancel{g_{\phi r,\phi}} + g_{\phi\phi,r} - \cancel{g_{\phi r,\phi}} \right) = \frac{1}{r}$$

$$\nabla^2 \Phi = \Phi^{;i}{}_{;i} = (g^{ij}\Phi_{,j})_{;i}$$

$$= (g^{rr}\Phi_{,r})_{;r} + (g^{\phi\phi}\Phi_{,\phi})_{;\phi}$$

$j = r$

$$= \frac{\partial}{\partial r}\left(\frac{\partial \Phi}{\partial r}\right) + \cancel{\Gamma^{r}_{jr}\Phi^{,j}} + \frac{\partial}{\partial \phi}\left(\frac{1}{r^2}\frac{\partial \Phi}{\partial \phi}\right) + \Gamma^{\phi}_{j\phi}\Phi^{,j}$$

$$\boxed{\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}}$$

$$g = \det g_{ij} = r^2$$

$$d^4x = \left|\frac{\partial x}{\partial x'}\right| d^4x' = \sqrt{g}\, d^4x'$$

$$\cancel{d^4x' = \det\left|\frac{\partial x'}{\partial x}\right| d^4x}$$

$r\,dr\,d\phi = dx\,dy$

$$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\eta_{\alpha\beta}$$

$$\det g = \left[\det\left|\frac{\partial x'}{\partial x}\right|\right]^2 \det \eta$$

$$\nabla^2 \Phi = \frac{1}{\sqrt{g}}\left(\sqrt{g}\cdot g^{ij}\Phi_{,i}\right)_{,j}$$

$$= \frac{1}{r}\left(r \cdot 1 \cdot \frac{\partial \Phi}{\partial r}\right)_{,i} + \frac{1}{r}\left(r \cdot \frac{1}{r^2}\Phi_{,\phi}\right)_{,\phi}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2}$$

$\Rightarrow$ self-adjoint form

Remember <u>exterior derivative</u>.

$$\tilde{d}\tilde{\omega} = (p+1)\,\omega_{[\alpha\beta\ldots\rho;\sigma]} \qquad p\text{-form.}$$

$$\tilde{d}\omega_\alpha = \omega_{\alpha;\beta} - \omega_{\beta;\alpha}$$

$$= \left(\omega_{\alpha,\beta} - \Gamma^\mu_{\alpha\beta}\,\omega_\mu\right) - \left(\omega_{\beta,\alpha} - \Gamma^\mu_{\beta\alpha}\,\omega_\mu\right)$$

$$= \underline{\underline{\omega_{\alpha,\beta} - \omega_{\beta,\alpha}}} + \cancel{\left(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}\right)}\,\omega_\mu$$

$$\tilde{d}\omega_{\alpha\beta} = \omega_{\alpha\beta;\gamma} + \omega_{\gamma\alpha;\beta} + \omega_{\beta\gamma;\alpha}$$

$$- \omega_{\beta\alpha;\gamma} - \omega_{\alpha\gamma;\beta} - \omega_{\gamma\beta;\alpha}$$

$$= (\text{partial derivatives})$$

$$- \underline{\underline{\Gamma^\mu_{\alpha\gamma}}}\,\omega_{\mu\beta} - \Gamma^\mu_{\beta\gamma}\,\omega_{\alpha\mu}$$

$$- \Gamma^\mu_{\gamma\beta}\,\omega_{\mu\alpha} - \Gamma^\mu_{\alpha\beta}\,\omega_{\gamma\mu}$$

$$- \Gamma^\mu_{\beta\alpha}\,\omega_{\mu\gamma} - \underline{\underline{\Gamma^\mu_{\gamma\alpha}}}\,\omega_{\beta\mu}$$

$$+ \ldots$$

B. Geroch "An expression involving tensor fields and a derivative operator ∇_a which is independent of the choice of derivative operator is called a "concomitant".

Let $w_{a \dots c}$ be antisymmetric in all indices. The expression $\nabla_{[m} w_{a \dots c]}$ is a concomitant.

(Lie derivative. $[u, v]$)

Two connections. Γ, Γ'

$(\nabla V)^{\mu}{}_{\nu}$, $(\nabla' V)^{\mu}{}_{\nu}$ both tensors.

$(\nabla V)^{\mu}{}_{\nu} = V^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\lambda\nu} V^{\lambda}$

$(\nabla' V)^{\mu}{}_{\nu} = V^{\mu}{}_{,\nu} + \Gamma'^{\mu}{}_{\lambda\nu} V^{\lambda}$

$(\nabla - \nabla')V = \text{tensor} = (\Gamma - \Gamma')^{\mu}{}_{\lambda\nu} V^{\lambda}$

Can always work with "coordinate" Γ plus additional tensor field torsion

Vector field: $V^\mu(x^\alpha)$

"constant": unchanging. $\nabla_\alpha V^\mu = 0$.

(in general, too strict.)

observers can only observe along path.

$$\cancel{\frac{dV^\mu}{d\lambda}} = \cancel{\frac{\partial V^\mu}{\partial \lambda}} \quad \boxed{\nabla_u V^\mu = u^\alpha \nabla_\alpha V^\mu = 0}$$

u^μ = direction $\frac{dx^\mu}{d\lambda}$

$$u^\alpha V^\mu{}_{;\alpha} = u^\alpha \left(\frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\sigma\alpha} V^\sigma \right) = 0.$$

$$\hookrightarrow \frac{dx^\alpha}{d\lambda} \frac{\partial V^\mu}{\partial x^\alpha} = \frac{dV^\mu}{d\lambda}.$$

$$\boxed{\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} V^\alpha u^\beta = 0} \quad \nabla_u \hat{V} = 0$$

"parallel transport" V along u

- can apply to $\hat{u}$ itself

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$

"Geodesic equation".

$u = \frac{dx}{d\lambda}$.

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

(As straight as can be)

Let go of $g_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \eta_{\mu\nu}$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

on sphere. $(r=a)$ $\boxed{ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2)}$

$(a=1)$ $\underline{g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}}$. $\cancel{\Gamma^\theta{}_{\theta\theta}}$ $\cancel{\Gamma^\theta{}_{\theta\phi} = \Gamma^\theta{}_{\phi\theta}}$ $\Gamma^\theta{}_{\phi\phi}$

$\cancel{\Gamma^\phi{}_{\theta\theta}}$ $\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta}$ $\cancel{\Gamma^\phi{}_{\phi\phi}}$

need 2 ϕ's $\quad g_{\phi\phi}$

need 1 θ $\quad \frac{\partial}{\partial\theta}$

$$\Gamma^\theta{}_{\phi\phi} = \frac{1}{2} g^{\theta\theta} (g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta})$$

$$= \frac{1}{2}(1)(-2\sin\theta\cos\theta) = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{1}{2} g^{\phi\phi} (\cancel{g_{\phi\theta,\phi}} + g_{\phi\phi,\theta} - \cancel{g_{\theta\phi,\phi}})$$

$$= \frac{1}{2} \frac{1}{\sin^2\theta} (2\sin\theta\cos\theta) = \frac{\cos\theta}{\sin\theta}$$

Sphere. $\theta(\lambda)$ $\phi(\lambda)$

$$\frac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta \cdot \left(\frac{d\phi}{d\lambda}\right)^2 = 0$$ $\mu=\theta$, $\alpha=\beta=\phi$

$$\frac{d^2\phi}{d\lambda^2} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$ $\mu=\phi$, $\alpha=\phi$ $\beta=\theta$, $\alpha=\theta$ $\beta=\phi$

$$\frac{d}{d\lambda}\left(\sin^2\theta \cdot \frac{d\phi}{d\lambda}\right) = 2\sin\theta\cos\theta\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} + \sin^2\theta \cdot \frac{d^2\phi}{d\lambda^2}$$

$$= \sin^2\theta\left(\frac{d^2\phi}{d\lambda^2} + 2\frac{\cos\theta}{\sin\theta}\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda}\right) = 0.$$

$$\boxed{\sin^2\theta \cdot \frac{d\phi}{d\lambda} = \alpha = \text{constant}}$$

$$\boxed{\frac{d^2\theta}{d\lambda^2} - \alpha^2 \cdot \frac{\cos\theta}{\sin^3\theta} = 0}$$

$$\frac{d}{d\lambda}\left(\frac{1}{2}\left(\frac{d\theta}{d\lambda}\right)^2 + \frac{1}{2}\frac{\alpha^2}{\sin^2\theta}\right)$$

$$= \frac{d\theta}{d\lambda} \cdot \frac{d^2\theta}{d\lambda^2} + \alpha^2\left(-\frac{\cos\theta}{\sin^3\theta}\right)\frac{d\theta}{d\lambda} = 0$$

$$\boxed{\frac{1}{2}\left(\frac{d\theta}{d\lambda}\right)^2 + \frac{1}{2}\frac{\alpha^2}{\sin^2\theta} = \frac{1}{2}\beta^2}$$

$\left(\frac{d\theta}{d\lambda}\right)^2 \geq 0$

$\sin^2\theta \leq 1$

$\beta^2 \geq \alpha^2$