9/30/2016

plane polar. $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$\boxed{\Gamma^r_{\phi\phi} = -r}$ $\boxed{\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}}$

constant $\hat{v}$. $\nabla_i v^j = 0$ $v^r_{;r} = 0$ $v^r_{;\phi} = 0$ $v^\phi_{;r} = 0$ $v^\phi_{;\phi} = 0$

$v^r_{;r} = v^r_{,r} + \cancel{\Gamma^r_{kr} v^k} = 0$ $\boxed{v^r_{,r} = 0}$

$v^\phi_{;r} = v^\phi_{,r} + \Gamma^\phi_{kr} v^k = v^\phi_{,r} + \Gamma^\phi_{\phi r} v^\phi = v^\phi_{,r} + \frac{1}{r} v^\phi = 0$

$\frac{\partial}{\partial r}(r v^\phi) = v^\phi + r \frac{\partial v^\phi}{\partial r} = r\left(\frac{1}{r} v^\phi + v^\phi_{,r}\right) = 0.$

$\boxed{v^r = f(\phi)}$ $\boxed{r v^\phi = g(\phi)}$ $v^{\hat\phi} = (g_{\phi\phi})^{1/2} v^\phi = g(\phi)$

$v^r_{;\phi} = v^r_{,\phi} + \Gamma^r_{k\phi} v^k = v^r_{,\phi} + \Gamma^r_{\phi\phi} v^\phi = v^r_{,\phi} - r v^\phi$

$\boxed{\frac{\partial f}{\partial \phi} = g}$

$v^\phi_{;\phi} = v^\phi_{,\phi} + \Gamma^\phi_{k\phi} v^k = v^\phi_{,\phi} + \Gamma^\phi_{r\phi} v^r = v^\phi_{,\phi} + \frac{1}{r} v^r$

$\frac{1}{r} \frac{\partial g}{\partial \phi} = -\frac{1}{r} f.$ $\boxed{\frac{\partial g}{\partial \phi} = -f}$

$\frac{\partial^2 f}{\partial \phi^2} = \frac{\partial g}{\partial \phi} = -f$

$\frac{\partial^2 g}{\partial \phi^2} = -\frac{\partial f}{\partial \phi} = -g.$

$f = A \cdot \cos\phi + B \cdot \sin\phi \rightarrow \cos(\phi - \phi_0)$

$g = \frac{\partial f}{\partial \phi} = -\sin(\phi - \phi_0)$

$\vec{v} = \hat{n}_0$ $\hat{n} \cdot \hat{e}_r = \cos(\phi - \phi_0)$ $\hat{n}.$

sphere. $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$

$\underline{v^\theta_{;\theta} = 0} = v^\theta_{,\theta} + \cancel{\Gamma^\theta_{k\theta} v^k} \qquad \boxed{\frac{\partial v^\theta}{\partial\theta} = 0} \qquad \boxed{v^\theta = f(\phi).}$

$v^\phi_{;\theta} = 0 = v^\phi_{,\theta} + \Gamma^\phi_{k\theta} v^k = v^\phi_{,\theta} + \Gamma^\phi_{\phi\theta} v^\phi$

$\frac{\partial v^\phi}{\partial\theta} + \frac{\cos\theta}{\sin\theta} v^\phi = 0 \qquad \frac{\partial}{\partial\theta}\left(\sin\theta\, v^\phi\right) = \cos\theta\, v^\phi + \sin\theta \frac{\partial v^\phi}{\partial\theta}$

$= \sin\theta\left(\frac{\partial v^\phi}{\partial\theta} + \frac{\cos\theta}{\sin\theta} v^\phi\right) = 0 \qquad \boxed{\sin\theta\, v^\phi = g(\phi)}$

$v^\theta_{;\phi} = v^\theta_{,\phi} + \Gamma^\theta_{k\phi} v^k = v^\theta_{,\phi} + \Gamma^\theta_{\phi\phi} v^\phi = v^\theta_{,\phi} - \sin\theta\cos\theta\, v^\phi$

$\frac{\partial v^\theta}{\partial\phi} = \boxed{\frac{\partial f}{\partial\phi} = \underline{\sin\theta}\cos\theta\, \underline{v^\phi} = \cos\theta \cdot g(\phi)}$

$v^\phi_{;\phi} = v^\phi_{,\phi} + \Gamma^\phi_{k\phi} v^k = v^\phi_{,\phi} + \Gamma^\phi_{r\phi} v^r$

$\frac{\partial v^\phi}{\partial\phi} = \frac{1}{\cancel{\sin\theta}} \cdot \frac{\partial g}{\partial\phi} = -\frac{\cos\theta}{\cancel{\sin\theta}} \cdot f$

$\boxed{\frac{\partial f}{\partial\phi} = \cos\theta\, g} \qquad \boxed{\frac{\partial g}{\partial\phi} = -\cos\theta \cdot f}$

Therefore

$\frac{\partial^2 f}{\partial\phi^2} = \cos\theta \cdot \frac{\partial g}{\partial\phi} = \cos\theta \cdot \left(-\cos\theta\, f\right) = -\cos^2\theta \cdot f$

Parallel transport : $\nabla_u \hat{v} = 0$

along $x^\mu(\lambda)$ $\quad u^\mu = \frac{dx^\mu}{d\lambda}$

$$\nabla_u \hat{v} = u^j v^i_{;j} = u^j \left(v^i_{,j} + \Gamma^i_{kj} v^k \right)$$

$$= \frac{dx^j}{d\lambda}\frac{\partial v^i}{\partial x^j} + \Gamma^i_{kj} v^k u^j = \boxed{\frac{dv^i}{d\lambda} + \Gamma^i_{jk} v^j u^k = 0}$$

$i = \theta$

$$\frac{dv^\theta}{d\lambda} + \Gamma^\theta_{jk} v^j u^k = \frac{dv^\theta}{d\lambda} + \Gamma^\theta_{\phi\phi} v^\phi u^\phi = 0$$

$$\boxed{\frac{dv^\theta}{d\lambda} + (-\sin\theta\cos\theta)\, v^\phi u^\phi = 0}$$

$i = \phi$

$$\frac{dv^\phi}{d\lambda} + \Gamma^\phi_{jk} v^j u^k = \frac{dv^\phi}{d\lambda} + \Gamma^\phi_{\phi\theta} \left(v^\phi u^\theta + v^\theta u^\phi \right) = 0$$

$$\boxed{\frac{dv^\phi}{d\lambda} + \frac{\cos\theta}{\sin\theta}\left(v^\phi u^\theta + v^\theta u^\phi \right) = 0}$$

$\theta = \theta_0 = \frac{\pi}{2}$ $\quad \frac{dv^\theta}{d\lambda} = 0 \quad \frac{dv^\phi}{d\lambda} = 0 .\quad v = \text{constant}$ ✓

$\phi = \phi_0$ $\quad \theta = \lambda$, $\quad u^\phi = 0 \quad u^\theta = 1$

$$\frac{dv^\theta}{d\theta} = 0$$

$$\boxed{\begin{aligned} v^\theta &= \text{constant} \\ v^\phi &= \text{const}/\sin\theta \end{aligned}}$$

$$\frac{dv^\phi}{d\theta} + \frac{\cos\theta}{\sin\theta}\left(v^\phi \cdot 1 \right) = 0$$

$$\frac{d}{d\theta}\left(\sin\theta\, v^\phi\right) = \cos\theta\, v^\phi + \sin\theta \frac{dv^\phi}{d\theta}$$

$$= \sin\theta \left(\frac{dv^\phi}{d\theta} + \frac{\cos\theta}{\sin\theta} v^\phi \right) = 0$$

Parallel transport of tangent $\nabla_u u = 0$

$$\frac{du^i}{d\lambda} + \Gamma^i_{jk} u^j u^k = 0$$

$$\frac{du^\theta}{d\lambda} - \sin\theta\cos\theta\,(u^\phi)^2 = 0$$

$$\frac{du^\phi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta} u^\theta u^\phi = 0$$

$$\frac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\phi}{d\lambda^2} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

$$\frac{d}{d\lambda}\left(\sin^2\theta\frac{d\phi}{d\lambda}\right) = 2\sin\theta\cos\theta\,\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} + \sin^2\theta\cdot\frac{d^2\phi}{d\lambda^2}$$

$$= \sin^2\theta\left(\frac{d^2\phi}{d\lambda^2} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda}\right) = 0$$

$$\boxed{\sin^2\theta\cdot\frac{d\phi}{d\lambda} = \alpha} = \text{constant} \qquad \frac{d\phi}{d\lambda} = \frac{\alpha}{\sin^2\theta}$$

$$\frac{du}{d\theta} - \sin\theta\cos\theta \cdot \left(\frac{\alpha}{\sin^2\theta}\right)^2 = 0.$$

$$\frac{d}{d\lambda}\left(\frac{1}{2}\left(\frac{d\theta}{d\lambda}\right)^2 + \frac{1}{2}\frac{\alpha^2}{\sin^2\theta}\right)$$

$$= \frac{d\theta}{d\lambda}\cdot\frac{d^2\theta}{d\lambda^2} + \alpha^2\left(-\frac{\cos\theta}{\sin^3\theta}\right)\frac{d\theta}{d\lambda} = 0$$

$$\boxed{\frac{1}{2}\left(\frac{d\theta}{d\lambda}\right)^2 + \frac{1}{2}\frac{\alpha^2}{\sin^2\theta} = \frac{1}{2}\beta^2} = \text{constant.}$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 \ge 0 . \quad \sin^2\theta \le 1. \qquad \beta^2 \ge \alpha^2$$

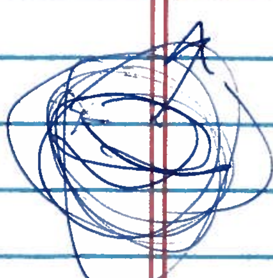
$\lambda \leftrightarrow \phi$ shape.

$$\frac{d\theta}{d\lambda} = \frac{d\theta}{d\phi}\cdot\frac{d\phi}{d\lambda} = \frac{d\theta}{d\phi}\frac{\alpha}{\sin^2\theta} = -\alpha\frac{d}{d\phi}(\cot\theta)$$

$$\frac{1}{2}\alpha^2\left[\frac{d}{d\phi}(\cot\theta)\right]^2 + \frac{1}{2}\alpha^2(1+\cot^2\theta) = \frac{1}{2}\beta^2$$

$$\left[\frac{d}{d\phi}(\cot\theta)\right]^2 + \left[\cot\theta\right]^2 = \left(\frac{\beta^2}{\alpha^2} - 1\right).$$

$$\boxed{\cot\theta = -\tan\theta_0 \cdot \cos(\phi-\phi_0)} \qquad \boxed{\tan^2\theta_0 = \frac{\beta^2}{\alpha^2} - 1}$$



$$\hat{n}\cdot\hat{n}_0 = 0 \qquad \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0) = 0$$