10/5/2016

Covariant derivative:

$$(\nabla V)^\alpha{}_\beta = \nabla_\beta V^\alpha = V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu$$

Parallel transport: $\nabla_u \hat{V} = 0$ $\quad \left(u = \frac{dx}{d\lambda} \right)$

$$u^\beta (V^\alpha{}_{;\beta}) = u^\beta (V^\alpha{}_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu)$$

$$= \frac{dx^\beta}{d\lambda} \cdot \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu u^\beta = \boxed{\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} V^\mu u^\nu = 0}$$

Geodesic: parallel transport of tangent.

$\nabla_u \hat{u} = 0$ $\qquad \boxed{\frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0}$

$$\nabla_u (\hat{V} \cdot \hat{V}) = 2\hat{V} \cdot (\nabla_u \hat{V}) = 0.$$

Norm preserved

$$\nabla_u (\hat{u} \cdot \hat{u}) = 2\hat{u} \cdot (\nabla_u \hat{u}) = 0$$

timelike
spacelike
null
preserved

$$\boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0}$$

$\lambda \to$ acts "Affine" parameter.

②

$$\nabla_u(\hat{u}\cdot\hat{v}) = \cancel{(\nabla_u\hat{u})}\cdot\hat{v} + \hat{u}\cdot\cancel{(\nabla_u\hat{v})} \Rightarrow 0$$

parallel transport along geodesic preserves angle. (~~[illegible]~~) sphere.

look at covariant version:

$$\nabla_u u_\alpha = u^\beta \nabla_\beta u_\alpha = u^\beta u_{\alpha;\beta} = \frac{du_\alpha}{d\lambda} - \Gamma^\mu_{\alpha\beta} u^\beta u_\mu$$

$$\boxed{\frac{du_\alpha}{d\lambda} = \Gamma_{\mu\alpha\nu} u^\mu u^\nu}$$

$$\underset{u^\mu u^\nu}{\Gamma_{\mu\alpha\nu}} = \frac{1}{2}\left(g_{\mu\alpha,\nu} + g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu}\right)u^\mu u^\nu$$

$$g_{\alpha\mu,\nu}u^\mu u^\nu - g_{\alpha\nu,\mu}u^\mu u^\nu \Rightarrow 0$$

$$\boxed{\frac{du_\alpha}{d\lambda} = \frac{1}{2} g_{\mu\nu,\alpha} u^\mu u^\nu}$$

$g_{\mu\nu}$ independent of x^α $\rightarrow$ $\boxed{u_\alpha = \text{constant}}$

$\frac{\partial g_{\mu\nu}}{\partial x^{\sigma*}} = 0$

Let $K^{\mu} = \delta^{\mu}_{\sigma*}$ $\quad \vec{K} = K^{\mu}\hat{e}_{\mu} = \hat{e}_{\sigma*} = \frac{\partial}{\partial x^{\sigma*}}$

$$u_{\sigma*} = K^{\mu}u_{\mu} = K_{\nu}u^{\nu}$$

Along geodesic $\quad \frac{du_{\sigma*}}{d\lambda} = \frac{d}{d\lambda}(K_{\nu}u^{\nu})$

$$= u^{\mu}\nabla_{\mu}(K_{\nu}u^{\nu})$$

$$\frac{du_{\sigma*}}{d\lambda} = u^{\mu}\left[\nabla_{\mu}K_{\nu}\cdot u^{\nu} + K_{\nu}\nabla_{\mu}u^{\nu}\right]$$

$$= (\nabla_{\mu}K_{\nu})u^{\mu}u^{\nu} + K_{\nu}\,\cancel{(u^{\mu}\nabla_{\mu}u^{\nu})}$$

geodesic

$$= \nabla_{(\mu}K_{\nu)}u^{\mu}u^{\nu} = 0.$$

Killing's Equation: $\nabla_{(\mu}K_{\nu)} = K_{(\mu;\nu)} = 0$

$$\boxed{\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0}$$

generalization, coordinate independent version of $\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = 0$.

$$K_{(\mu;\nu)} = 0$$

$$\nabla_u (K \cdot u) = u^\mu \nabla_\mu (K_\nu u^\nu)$$

$$= u^\mu (K_{\mu;\nu} u^\nu + K_\nu u^\nu{}_{;\mu})$$

$$= \cancel{K_{\mu;\nu} u^\mu u^\nu} + K_\nu (\cancel{u^\mu u^\nu{}_{;\mu}})$$

$$K_{(\mu;\nu)} = 0 \qquad \nabla_u (u) = 0$$

$\boxed{K \cdot u = \text{constant}}$ along geodesic.

$$J^\mu = T^{\mu\nu} K_\nu$$

$$J^\mu{}_{;\mu} = (T^{\mu\nu} K_\nu)_{;\mu} = \cancel{T^{\mu\nu}{}_{;\mu}} K_\nu + \cancel{T^{\mu\nu} K_{\mu;\nu}}$$

$$\partial_\alpha \partial_\beta f - \partial_\beta \partial_\alpha f = f_{,\beta\alpha} - f_{,\alpha\beta} = 0.$$

$$\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha f$$

$$= (\cancel{f_{,\beta\alpha}} - \Gamma^\mu_{\beta\alpha} f_{,\mu}) - (\cancel{f_{,\alpha\beta}} - \Gamma^\mu_{\alpha\beta} f_{,\mu})$$

$$= (\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) f_{,\mu} = 0. \quad \text{(w/o torsion)}$$

$$\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$$

$$T^{\mu\ \alpha}_{\ \ \beta}$$

$$= \nabla_\alpha (V^\mu_{,\beta} + \Gamma^\mu_{\lambda\beta} V^\lambda) - \nabla_\beta (V^\mu_{,\alpha} + \Gamma^\mu_{\lambda\alpha} V^\lambda)$$

$$= (\cancel{V^\mu_{,\beta}} + \Gamma^\mu_{\lambda\beta} V^\lambda)_{,\alpha} + \Gamma^\mu_{\nu\alpha} (V^\nu_{,\beta} + \Gamma^\nu_{\lambda\beta} V^\lambda) - \cancel{\Gamma^\rho_{\beta\alpha} (V^\mu_{,\rho} + \Gamma^\mu_{\lambda\rho} V^\lambda)}$$

$$- (\cancel{V^\mu_{,\alpha}} + \Gamma^\mu_{\lambda\alpha} V^\lambda)_{,\beta} + \Gamma^\mu_{\nu\beta} (V^\nu_{,\alpha} + \Gamma^\nu_{\lambda\alpha} V^\lambda) - \cancel{\Gamma^\rho_{\alpha\beta} (V^\mu_{,\rho} + \Gamma^\mu_{\lambda\rho} V^\lambda)}$$

$$= \underline{V^\lambda_{,\alpha} \Gamma^\mu_{\lambda\beta}} + V^\lambda \Gamma^\mu_{\lambda\beta,\alpha} + V^\nu_{,\beta} \cancel{\Gamma^\mu_{\nu\alpha}} + \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\lambda\beta} V^\lambda$$

$$- V^\lambda_{,\beta} \cancel{\Gamma^\mu_{\lambda\alpha}} - V^\lambda \Gamma^\mu_{\lambda\alpha,\beta} - \underline{V^\nu_{,\alpha}} \Gamma^\mu_{\nu\beta} - \Gamma^\mu_{\nu\beta} \Gamma^\nu_{\lambda\alpha} V^\lambda$$

$$= \left[-\Gamma^\mu_{\lambda\alpha,\beta} + \Gamma^\mu_{\lambda\beta,\alpha} + \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\nu\beta} \Gamma^\nu_{\lambda\alpha} \right] V^\lambda$$

$$= R^\mu{}_{\lambda\alpha\beta} V^\lambda \quad \blacksquare.$$

$$\boxed{\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = R^\mu{}_{\lambda\alpha\beta} V^\lambda} \quad (3.112)$$

$$R^\mu{}_{\lambda\alpha\beta} = -\Gamma^\mu{}_{\lambda\alpha,\beta} + \Gamma^\mu{}_{\lambda\beta,\alpha} + \Gamma^\mu{}_{\nu\alpha}\Gamma^\nu{}_{\lambda\beta} - \Gamma^\mu{}_{\nu\beta}\Gamma^\nu{}_{\lambda\alpha}$$

(3.113)

Symmetries: $R_{\mu\lambda\alpha\beta} = g_{\mu\nu} R^\nu{}_{\lambda\alpha\beta}$.

Frame ($\Gamma = 0$)

$$R_{\mu\lambda\alpha\beta} = g_{\mu\nu}\left(-\Gamma^\nu{}_{\lambda\alpha,\beta} + \Gamma^\nu{}_{\lambda\beta,\alpha}\right)$$

$$= g_{\mu\nu}\Big[\left(\left(-\tfrac{1}{2}g^{\nu\gamma}\right)\left(g_{\gamma\lambda,\alpha} + g_{\gamma\alpha,\lambda} - g_{\alpha\lambda,\gamma}\right)\right)_{,\beta}$$

$$+\left(\left(\tfrac{1}{2}g^{\nu\gamma}\right)\left(g_{\gamma\lambda,\beta} + g_{\gamma\beta,\lambda} - g_{\lambda\beta,\gamma}\right)\right)_{,\alpha}\Big]$$

$$\rightarrow -\tfrac{1}{2}\,\delta_\mu^\gamma \Big(+\cancel{g_{\gamma\lambda,\alpha\beta}} + g_{\gamma\alpha,\lambda\beta} - g_{\alpha\lambda,\gamma\beta}$$

$$-\cancel{g_{\gamma\lambda,\beta\alpha}} - g_{\gamma\beta,\lambda\alpha} + g_{\lambda\beta,\gamma\alpha}\Big)$$

$$\boxed{= \tfrac{1}{2}\left(g_{\alpha\lambda,\beta\mu} + g_{\beta\mu,\alpha\lambda} - g_{\alpha\mu,\beta\lambda} - g_{\beta\lambda,\alpha\mu}\right)}$$

Symmetries.

$$\boxed{R_{\mu\lambda\alpha\beta} = R_{\mu\lambda[\alpha\beta]}} \qquad (3.130)$$

$$\boxed{R_{\mu\lambda\alpha\beta} = R_{[\mu\lambda]\alpha\beta}.} \qquad (3.129)$$

$$\boxed{R_{\mu\lambda\alpha\beta} = R_{\alpha\beta\mu\lambda}}$$

"left to your imagination"

$$R_{\mu\lambda\alpha\beta} + R_{\mu\alpha\beta\lambda} + R_{\mu\beta\lambda\alpha} = 0$$

$$\boxed{R_{\mu[\lambda\alpha\beta]} = 0}$$

$$\frac{\left[\frac{1}{2}n(n-1)\right] \times \left[\frac{1}{2}n(n-1)+1\right]}{2} = \frac{1}{8}\left(n^4 - 2n^3 + 3n^2 - 2n\right)$$

$n=1$ $R=0$.

$n=2$ R_{0101} 1

$n=3$ $\frac{3\times4}{2} = 6.$

$n=4$ $\frac{10\times11}{2}$

$$\boxed{\frac{1}{12}n^2(n^2-1)}$$

$n=1$ 0

$n=2$ $\frac{1}{12}\cdot4\cdot3 = 1$

$n=3$ $\frac{1}{12}\cdot9\cdot8 = 6$

$n=4$ $\frac{1}{12}\cdot16\cdot15$

$= 4\cdot5 = 20$