10/12/2016

Know: $\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = R^\mu{}_{\lambda\alpha\beta} V^\lambda$ (3.112)

Know: $\nabla_u \nabla_v \Phi - \nabla_v \nabla_u \Phi = \nabla_{[u,v]} \Phi$ (HW)

$$\Rightarrow \quad \nabla_u \nabla_v w^\mu - \nabla_v \nabla_u w^\mu = \nabla_{[u,v]} w^\mu + R^\mu{}_{\alpha\beta\gamma} \cancel{u^\alpha v^\beta w^\gamma} \; w^\alpha u^\beta v^\gamma$$

R: no ∂u, ∂v

$[u,v]$ no $\partial \Gamma$

geodesic $\nabla_u \hat{u} = 0$

$\hat{u} = \frac{d}{d\lambda}$ $\qquad \nabla_u \hat{V} = \frac{dx^\alpha}{d\lambda} \frac{\partial V^\mu}{\partial x^\alpha} = \frac{dV^\mu}{d\lambda}$

$\hat{n} = \frac{d}{dn}$ $\qquad \nabla_n \hat{V} = \frac{dx^\alpha}{dn} \frac{\partial V^\mu}{\partial x^\alpha} = \frac{dV^\mu}{dn}$

$$[\hat{u}, \hat{n}] = \nabla_u \hat{n} - \nabla_n \hat{u}$$

$$= u^\alpha \left(n^\mu{}_{,\alpha} + \Gamma^\mu_{\lambda\alpha} n^\lambda \right) - n^\lambda \left(u^\mu{}_{,\lambda} + \cancel{\Gamma^\mu_{\alpha\lambda} u^\alpha} \right)$$

$$= \underbrace{\frac{\partial n^\mu}{\partial \lambda}}_{\to 0} - \underbrace{\frac{\partial u^\mu}{\partial n}}_{\to 0}$$

(n constructed $\perp$ u. $\nabla_n u = 0$.)

$$\boxed{[u, n] = 0}$$

$$\nabla_u \nabla_u \hat{n} = \nabla_u (\nabla_n \hat{u})$$

$$= \cancel{\nabla_n (\nabla_u \hat{u})} + (\nabla_u \nabla_n \hat{u} - \nabla_n \nabla_u \hat{u})$$

$$= [\nabla_u, \nabla_n] \hat{u}$$

$$= \cancel{\nabla_{[u,n]}} \hat{u} + R^\mu{}_{\alpha\beta\gamma} u^\alpha u^\beta n^\gamma$$

$$\boxed{\nabla_u \nabla_u n^\mu = - R^\mu{}_{\alpha\beta\gamma} u^\alpha n^\beta u^\gamma}$$

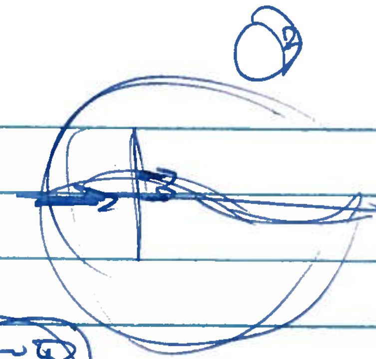
geodesic deviation:

" ∇_u " $\rightarrow$ " $\frac{D}{D\tau}$ " $\frac{D}{D\tau}$

- commutators
- parallel transport
- geodesic deviation

$\partial \Gamma$.

Near equator of sphere



$$ds^2 = a^2 (d\theta^2 + \sin^2\phi \, d\phi^2) \qquad \theta = \frac{\pi}{2}$$

$$\Gamma^i_{jk} = \cancel{\frac{\cos\theta}{\sin\theta}} , \cancel{\sin\theta\cos\theta} = 0.$$

$$\boxed{R^{\theta\phi}{}_{\theta\phi} = \frac{1}{a^2}} \Leftarrow$$

Equatorial geodesic: $\theta(\lambda), \phi(\lambda)$.

$@ \lambda = 0 . \; u^i_0 = \begin{pmatrix} d\theta/d\lambda \\ d\phi/d\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\frac{du^i}{d\lambda} + \Gamma^i_{jk} u^j u^k = 0 \qquad \frac{du^i}{d\lambda} = 0 \qquad u^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ all } \lambda.$$

$$\theta = \frac{\pi}{2} \qquad \phi = \lambda .$$

$$\frac{d^2 n^i}{d\lambda^2} = - R^i{}_{jkl} u^j n^k u^l = - R^i{}_{\phi k \phi} n^k$$

$$= - R^i{}_{\phi\theta\phi} n^\theta$$

$$\frac{d^2 \Delta\phi}{d\lambda^2} = 0 \qquad \frac{d\Delta\phi}{d\lambda} = \text{constant} = 0 \qquad \left.\frac{dn^i}{d\lambda}\right|_0 = 0 \qquad n^i(0) = \begin{pmatrix} \Delta\theta \\ 0 \end{pmatrix}$$

$\Delta\phi = \text{constant}$

$$\frac{d^2 \Delta\theta}{d\lambda^2} = - \Delta\theta \qquad \Delta\theta(\lambda) = (\Delta\theta)_0 \cos\lambda$$

$$\boxed{\Delta\theta = (\Delta\theta)_0 \cos\phi}$$

("attraction"??)

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Change coordinates

$$V'^{\alpha} = \left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\right) V^{\beta}$$

$$\boxed{g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu}}$$

$$x'^{\mu} = a^{\mu} + b^{\mu}_{\alpha} x^{\alpha} + c^{\mu}_{(\alpha\beta)} x^{\alpha} x^{\beta} + d^{\mu}_{(\alpha\beta\gamma)} x^{\alpha} x^{\beta} x^{\gamma} + \dots$$

b^{μ}_{α} $\quad 4\times4 = 16$ free parameters $\qquad g'_{\alpha\beta} = \eta_{\alpha\beta}$ $\quad \frac{4\cdot5}{2} = 10$ conditions

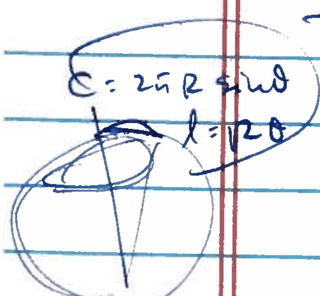
$c^{\mu}_{(\alpha\beta)}$ $\quad 4\cdot10 = 40$ Free parameters. $\qquad g'_{\alpha\beta,\gamma}$ $\quad \Gamma^{\alpha}_{\beta\gamma} \to 0$ $\quad 4\times10 = 40$ conditions

"local freely falling frame"

$$d^{\mu}_{(\alpha\beta\gamma)} = 4\cdot\left(\frac{4\cdot5\cdot6}{3!}\right) = 80 \text{ parameters.}$$

$g'_{\alpha\beta,\gamma\delta}$. $\quad (10)(10) = 100$ quantities

20 d.f. can't be zeroed. $\quad \frac{1}{12} n^2(n^2-1) = 20$



$$\lim_{l\to0} \frac{6}{l^2}\left(1 - \frac{C}{2\pi l}\right) = \frac{1}{R^2}$$

(5)

Showed for $\Gamma \to 0$.

$$R_{\rho\sigma\mu\nu} = \frac{1}{2}\left(g_{\rho\nu,\sigma\mu} - g_{\sigma\nu,\rho\mu} - g_{\rho\mu,\sigma\nu} + g_{\sigma\mu,\rho\nu}\right)$$
$$+ \cancel{\Gamma^2} \quad (\Gamma \to 0)$$

$$\nabla_\lambda R_{\rho\sigma\mu\nu} = \underline{\frac{1}{2}(\cdots)_{,\lambda}} + \cancel{\partial_\lambda(\Gamma^2)} + \cancel{\Gamma(\underline{\quad})}$$

Cycle last three indices. $\mu \to \nu \to \lambda \to \mu$.

$$R_{\rho\sigma\mu\nu;\lambda} + R_{\rho\sigma\nu\lambda;\mu} + R_{\rho\sigma\lambda\mu;\nu}$$

$$= \frac{1}{2}\Big(\cancel{g_{\rho\nu,\sigma\mu\lambda}} - g_{\sigma\nu,\rho\mu\lambda} - g_{\rho\mu,\sigma\nu\lambda} + g_{\sigma\mu,\rho\nu\lambda}$$

$$+ g_{\rho\lambda,\sigma\nu\mu} - g_{\sigma\lambda,\rho\nu\mu} - g_{\rho\nu,\sigma\lambda\mu} + g_{\sigma\nu,\rho\lambda\mu}$$

$$+ g_{\rho\mu,\sigma\lambda\nu} - g_{\sigma\mu,\rho\lambda\nu} - g_{\rho\lambda,\sigma\mu\nu} + g_{\sigma\lambda,\rho\mu\nu}\Big)$$

$$\boxed{R_{\rho\sigma[\lambda\mu;\nu]} = 0}$$

"Bianchi identity"

(Expressed as tensor, true in one frame $\to$ true in all)

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$$

Weyl $C_{\rho\sigma\mu\nu}$,

$$R_{\rho\sigma\mu\nu} = \frac{2}{n-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right)$$

$$- \frac{2}{(n-1)(n-2)} R \left(g_{\rho[\mu}g_{\nu]\sigma}\right)$$

$$+ C_{\rho\mu\nu\sigma}$$

Weyl curvature tensor

(All traces vanish)

Contract Bianchi identity:

$$g^{\rho\mu}\left(R_{\rho\sigma\mu\nu;\lambda} + R_{\rho\sigma\nu\lambda;\mu} + R_{\rho\sigma\lambda\mu;\nu}\right) = 0$$

$$R_{\sigma\nu;\lambda} + R^{\mu}{}_{\sigma\nu\lambda;\mu} + \left(-R_{\sigma\lambda;\nu}\right) = 0$$

Again: $R^{\lambda}{}_{\nu;\lambda} + R^{\mu}{}_{\nu;\mu} - R_{;\nu} = 0$

$$\boxed{R^{\mu}{}_{\nu;\mu} - \frac{1}{2}R_{;\nu} = 0}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$ Einstein tensor

$$G^{\mu}{}_{\mu} = G = R^{\mu}{}_{\mu} - \frac{1}{2} g^{\mu}{}_{\mu} R = R - \frac{1}{2}(4)R = -R$$

$\boxed{G \text{ is trace-reversed } R}$

$$G^{\mu}{}_{\nu;\mu} = \left(R^{\mu}{}_{\nu} - \frac{1}{2} g^{\mu}{}_{\nu} R\right)_{;\mu}$$

$$= R^{\mu}{}_{\nu;\mu} - \frac{1}{2} \delta^{\mu}{}_{\nu} R_{;\mu}$$

$$= \left(\frac{1}{2} R_{;\nu}\right) - \frac{1}{2} R_{;\nu} \quad = 0$$

$$\boxed{G^{\mu\nu}{}_{;\nu} = 0}$$

Gravity = geometry?

Everybody has same motion in gravitational field.

$$m\frac{d^2\vec{x}}{dt^2} = -m\vec{\nabla}\phi$$

$$\hookrightarrow \quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

$g_{\mu\nu} = \eta_{\mu\nu}$ somewhere,
small deviations in neighborhood $\rightarrow \quad g_{00} = -(1+2\phi)$

$$\frac{dx^\mu}{d\tau} = \begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} + O(v^2) \qquad (u^0 \gg u^i)$$

$$v = \frac{30\,\text{km/s}}{300{,}000\,\text{km/s}} = \underline{10^{-4}}.$$

$$d\tau = \gamma\, dt \simeq dt. \qquad \frac{\partial}{\partial t} \ll \frac{\partial}{\partial x}$$

$$\frac{d^2x^i}{dt^2} \simeq -\Gamma^i{}_{00}(1)^2 \qquad = \tfrac{1}{2}(2g_{i0,0} - g_{00,i})$$

$$\hookrightarrow \quad \Gamma^i{}_{00} = \tfrac{1}{2}g^{i\mu}(\cancel{g_{\mu 0,0}} + \cancel{g_{\mu 0,0}} - g_{00,\mu})$$

$$= \tfrac{1}{2}\frac{\partial}{\partial x^i}(1+2\phi) = -\nabla_i\phi.$$

$$\frac{d^2\vec{x}}{dt^2} \simeq -\vec{\nabla}\phi$$