10/19/2016

Relativistic gravity $\frac{d^2 \vec{x}}{dt^2} = -\vec{\nabla}\phi$.

metric, geodesic. $\hookrightarrow$ $g_{00} = -(1+2\phi)$

$$\boxed{\frac{d^2x^\mu}{d\tau^2} = -\frac{\partial\phi}{\partial x^\nu}\left(g^{\mu\nu} + u^\mu u^\nu\right)}$$

$a_\mu a^\mu = 0$ ✓

$\hookrightarrow$ $\boxed{g = e^{2\phi}\cdot\eta}$

$$S = \int d^4x \; \frac{1}{4\pi G} \, \eta^{\mu\nu} \partial_\mu\phi \cdot \partial_\nu\phi$$

$$- m \int d\tau \left(-e^{2\phi}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2}$$

$$\hookrightarrow \int d^4x \int d\tau \; \delta^{(4)}(x - x^\mu(\tau)) \left(m u^\mu u^\nu\right) \eta_{\mu\nu} e^{2\phi}$$

$$\left(-e^{2\phi}\eta_{\rho\sigma}u^\rho u^\sigma\right)^{1/2}$$

$g_{\mu\nu}$

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$\frac{\partial f}{\partial \phi} \to \int d\tau \; m u^\mu u^\nu \, \delta^{(4)}(x - x^\mu(\tau)) \; \eta_{\mu\nu} e^{2\phi})$

$= T^\mu{}_\mu$

$$-\frac{1}{4\pi G}\partial_\mu \partial^\mu \phi = T^\mu{}_\mu \qquad \boxed{\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = 4\pi G(\rho - 3p)}$$

$\nabla^2 \to \Box^2$ | $\rho \to \rho - 3p = T^\mu{}_\mu$

$\times$ ~~BB~~ $= 0$

Sensible. | Nordström 1913

$\times \cdot (T^\mu{}_\mu)_{EM} = 0$.

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Try again tensor theory.

Need. tensor made of $g_{\mu\nu}$, up to second derivatives

$R_{\mu\nu\rho\sigma}$. $R_{\mu\nu}$. $g_{\mu\nu}$ $\Longrightarrow$ $G_{\mu\nu}$.

$\nabla^2\phi = 4\pi G\rho$. $\rightarrow$ $(R_{\mu\nu}, G_{\mu\nu}, g_{\mu\nu}) = (\;)\,T_{\mu\nu}$.

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{[c^2][c^4]}\,T_{\mu\nu}}$$

Bonus: $G_{\mu\nu;\nu} = 0$ $\rightarrow$ $T^{\mu\nu}{}_{;\nu} = 0$

$+\Lambda g_{\mu\nu}$. cosmological constant.

weak field + photons $\rightarrow$ $\Gamma^{\mu}_{\alpha\beta}$

Einstein 1915

wikipedia. $\frac{8\pi G T}{c^4}$??

$\frac{30\ \text{km/s}}{300{,}000\ \text{km/s}} = 10^{-4}$

Weak field. $\quad g_{\mu\nu} = \eta_{\mu\nu} + \underline{h_{\mu\nu}} \qquad |h| \ll 1$

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} \left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma} \right)$$

$$\rightarrow \frac{1}{2} \eta^{\mu\gamma} \left(h_{\gamma\alpha,\beta} + h_{\gamma\beta,\alpha} - h_{\alpha\beta,\gamma} \right).$$

$$\Gamma = O(h) \qquad \cancel{\Gamma^2 = O(h^2)}$$

$$R_{\rho\sigma\mu\nu} = \frac{1}{2} \left(h_{\rho\nu,\sigma\mu} - h_{\sigma\nu,\rho\mu} - h_{\rho\mu,\sigma\nu} + h_{\sigma\mu,\rho\nu} \right)$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \eta^{\lambda\rho} R_{\lambda\mu\rho\nu}$$

$\rho \to \lambda$
$\sigma \to \mu$
$\mu \to \rho$
$\nu \to \nu$

$$= \eta^{\lambda\rho} \frac{1}{2} \left(h_{\lambda\nu,\rho\mu} - h_{\mu\nu,\lambda\rho} - h_{\lambda\rho,\mu\nu} + h_{\rho\mu,\lambda\nu} \right)$$

$$= \frac{1}{2} \left(h_{\nu\lambda,\mu}{}^{\lambda} - h_{\mu\nu,\lambda}{}^{\lambda} - h_{\lambda}{}^{\lambda}{}_{,\mu\nu} + h_{\mu\lambda,\nu}{}^{\lambda} \right)$$

$$R = R^{\mu}{}_{\mu} = \eta^{\mu\nu} R_{\mu\nu}$$

$$= \frac{1}{2} \left(h_{\mu\lambda}{}^{,\mu\lambda} - h^{\mu}{}_{\mu,\lambda}{}^{\lambda} - h_{\lambda}{}^{\lambda}{}_{,\mu}{}^{\mu} + h_{\mu\lambda}{}^{,\mu\lambda} \right)$$

$$R = h_{\lambda\mu}{}^{,\lambda\mu} - h^{\mu}{}_{\mu,\lambda}{}^{\lambda}.$$

Trace Reverse. $\boxed{\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h}$ $\quad h = h^{\alpha}{}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta}$

$$\to \quad G_{\alpha\beta} = \frac{1}{2}\left(\bar{h}_{\alpha\beta,\mu}{}^{\mu} + \cancel{\eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu}} - \cancel{\bar{h}_{\alpha\mu,\beta}{}^{\mu}} - \cancel{\bar{h}_{\beta\mu,\alpha}{}^{\mu}}\right)$$

Gauge choices $\quad$ E.M. $A_{\mu}{}^{,\mu} = 0 \longleftrightarrow \bar{h}_{\alpha\mu}{}^{,\mu} = 0$

Lorentz gauge.

What is "gauge transformation"?

Infinitesimal coordinate transformation

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha} \qquad \xi^{\alpha} = O(h).$$

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu} \qquad x^{\mu} = x'^{\mu} - \xi^{\mu}$$

$$\frac{\partial x^{\mu}}{\partial x'^{\alpha}} = \delta^{\mu}_{\alpha} - \frac{\partial \xi^{\mu}}{\partial x^{\alpha}}.$$

$$= \left(\delta^{\mu}_{\alpha} - \xi^{\mu}{}_{,\alpha}\right)\left(\delta^{\nu}_{\beta} - \xi^{\nu}{}_{,\beta}\right)\left(\eta_{\mu\nu} + h_{\mu\nu}\right)$$

$$= \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

$$\boxed{h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}}$$

$\uparrow$

(Killing vector $\to$ h unchanged)

(5)

$$h' = h'^{\alpha}{}_{\alpha} = h^{\alpha}{}_{\alpha} - 2\xi_{\alpha,}{}^{\alpha} = \underline{h - 2\xi^{\alpha}{}_{,\alpha}}$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} \cancel{\ldots} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha}$$

<u>why?</u>

$$\bar{h}'^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}{}_{,\nu} - \xi^{\nu,\mu}{}_{,\nu} + \eta^{\mu\nu}\xi^{\alpha}{}_{,\alpha\nu}$$

$$\bar{h}'^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \Box^2\xi^{\mu}$$

want $\underline{\bar{h}'^{\mu\nu}{}_{,\nu} = 0}$ $\Rightarrow$ $\underline{\Box^2\xi^{\mu} = \bar{h}^{\mu\nu}{}_{,\nu}}$ know how to solve (know G.)

Given $\bar{h}$, can find $\underline{\bar{h}'}$ (gauge X-form) such that $\bar{h}^{\mu\nu}{}_{,\nu} = 0$
(still have freedom to add $\Box^2\xi^{\mu} = 0$)

$$G_{\mu\nu} = -\frac{1}{2}\Box^2\bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\boxed{\Box^2\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$$ linearized GR

<u>MTW (18.8b)</u>

c.f. $$\boxed{\Box^2 A^{\mu} = -4\pi J^{\mu}}$$

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N.R. sources: $T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \simeq \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$p \ll \rho$

$\rightarrow$ only $\bar{h}_{00}$ is nonzero

$\nabla^2\phi = 4\pi G\rho \qquad \nabla^2 \bar{h}_{00} = -16\pi G T_{00} = -16\pi G\rho \qquad \boxed{\bar{h}_{00} = -4\phi}$

$$\bar{h} = \bar{h}^{\mu}{}_{\mu} = \bar{h}^{0}{}_{0} = +4\phi.$$

$$h = \begin{pmatrix} -4\phi & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}(4\phi)$$

$$= \begin{pmatrix} -2\phi & & & \\ & -2\phi & & \\ & & -2\phi & \\ & & & -2\phi \end{pmatrix}$$

$$g_{\mu\nu} = \left(\begin{array}{c|c} -(1+2\phi) & \\ \hline & (1-2\phi)\,\delta_{ij} \end{array}\right) \qquad \text{Z.E.9}$$

Nordstrom $\quad e^{2\phi}\eta_{\mu\nu} = \left(\begin{array}{c|c} -(1+2\phi) & \\ \hline & (1+2\phi)\,\delta_{ij} \end{array}\right)$

write: $ds^2 = -(1+2\phi)dt^2 + (1-2\gamma\phi)\,d\vec{x}^2$

$\uparrow$ PPN

GR $\gamma = 1$ $\qquad$ Nordstrom $\gamma = -1$ $\qquad$ Newton $\gamma = 0$ $\qquad$ Brans Dicke $\dfrac{1+\omega}{2+\omega}$

$\omega \to \infty$