10/24/2016 **Spherical symmetry:**

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2$$

6 diag

$$R^{tr}{}_{tr} = e^{-2\beta}\left[-\alpha_{,rr} - (\alpha_{,r})^2 + \alpha_{,r}\beta_{,r}\right]$$

$$R^{t\theta}{}_{t\theta} = R^{t\phi}{}_{t\phi} = -\frac{1}{r} e^{-2\beta} \alpha_{,r}$$

$$R^{r\theta}{}_{r\theta} = R^{r\phi}{}_{r\phi} = \frac{1}{r} e^{-2\beta} \beta_{,r}$$

$$R^{\theta\phi}{}_{\theta\phi} = \frac{1}{r^2}\left(1 - e^{-2\beta}\right) \qquad (5.13)$$

$$R^{t\theta}{}_{r\theta} = -\frac{1}{r} e^{-2\beta} \alpha_{,r} = R^{t\phi}{}_{r\phi}.$$

Vanishing: $tr\,t\theta$ $\quad tr\,t\phi$ $\quad tr\,r\theta$ $\quad tr\,r\phi$

$t\theta\,\theta\phi$ $\quad t\phi\,\theta\phi$ $\quad r\theta\,\theta\phi$ $\quad r\phi\,\theta\phi$

$t\phi\,r\theta$ $\quad tr\,\theta\phi$ $\quad t\theta\,r\phi$ $\quad t\theta\,t\phi$ $\quad r\theta\,r\phi$

②

$$R^t{}_t = R^{\mu t}{}_{\mu t} = R^{rt}{}_{rt} + R^{\theta t}{}_{\theta t} + R^{\phi t}{}_{\phi t}$$

$$= e^{-2\beta}\left(-\alpha_{,rr} - (\alpha_{,r})^2 + \alpha_{,r}\beta_{,r} - \frac{2}{r}\alpha_{,r}\right)$$

$$R^r{}_r = R^{rt}{}_{rt} + R^{r\theta}{}_{r\theta} + R^{r\phi}{}_{r\phi}$$

$$= e^{-2\beta}\left(-\alpha_{,rr} - (\alpha_{,r})^2 + \alpha_{,r}\beta_{,r} + \frac{2}{r}\beta_{,r}\right)$$

$$R^\theta{}_\theta = R^{\theta t}{}_{\theta t} + R^{\theta r}{}_{\theta r} + R^{\theta\phi}{}_{\theta\phi}$$

$$= e^{-2\beta}\left(-\frac{1}{r}\alpha_{,r} + \frac{1}{r}\beta_{,r}\right) + \frac{1}{r^2}\left(1-e^{-2\beta}\right) = R^\phi{}_\phi$$

$$\boxed{R^\phi{}_\phi = R^\theta{}_\theta = \frac{1}{r}e^{-2\beta}(\beta_{,r} - \alpha_{,r}) + \frac{1}{r^2}\left(1-e^{-2\beta}\right)} \quad (5.14)$$

Vacuum: $T_{\mu\nu} = 0$ (except $r=0$)

$R = 0$ $\qquad R^t{}_t - R^r{}_r = 0$

$$= e^{-2\beta}\left(-\alpha_{,rr} - (\alpha_{,r})^2 + \alpha_{,r}\beta_{,r} - \frac{2}{r}\alpha_{,r} + \alpha_{,rr} + (\alpha_{,r})^2 - \alpha_{,r}\beta_{,r} - \frac{2}{r}\beta_{,r}\right)$$

$$\frac{2}{r}\frac{\partial}{\partial r}(\alpha+\beta) = 0 \qquad \alpha+\beta = \text{constant} = 0 \quad (\text{scale})$$

Vacuum $\to \underline{T=0} \to G=0 \to R=0$.

$\boxed{\alpha+\beta=0}$ $\boxed{\beta=-\alpha}$

$$R^{\theta}{}_{\theta} = e^{2\alpha}\left(\frac{-2\alpha_{,r}}{r}\right) + \frac{1}{r^2}\left(1-e^{2\alpha}\right) = 0$$

$$\underline{\left(1-e^{2\alpha}\right) - 2r\,\alpha_{,r}\,e^{2\alpha} = 0}$$

$$\underline{\frac{\partial}{\partial r}\left[r\left(1-e^{2\alpha}\right)\right] = 0}$$

$$r\left(1-e^{2\alpha}\right) = \text{constant} = \underline{2m} \quad (r\to\infty)$$

$$\left(\frac{2Gm}{c^2}\right)$$

$$e^{2\alpha} = 1 - \frac{2m}{r}$$

$$\boxed{ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2\,d\Omega^2}$$

Schwarzschild. (1916). (12/22/15 letter)

Father of Martin Schwarzschild.

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\, u^\alpha u^\beta = 0,$$ But. $\theta = \frac{\pi}{2}$ $p_\theta = 0$

$$\frac{\partial g}{\partial t} = 0 \rightarrow p_t = \text{constant} = -m\tilde{E}$$ $E = m\tilde{E}$ = energy at ∞

$$\frac{\partial g}{\partial \phi} = 0 \rightarrow p_\phi = \text{constant} = m\tilde{L}$$ $L = m\tilde{L}$ = angular momentum at ∞.

$$p^t = m\frac{dt}{d\tau} = -\left(1 - \frac{2m}{r}\right)^{-1} \cdot m\tilde{E}$$

$$p^\phi = m\frac{d\phi}{d\tau} = \left(\frac{1}{r^2}\right) m\tilde{L} \qquad (\sin\theta = 1)$$

$$p^2 = p \cdot p = -\left(1 - \frac{2m}{r}\right)\left(\frac{m\,dt}{d\tau}\right)^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}\left(\frac{m\,dr}{d\tau}\right)^2 = -m^2$$

$$+ r^2 \cancel{\left(m\frac{d\theta}{d\tau}\right)^2} + r^2\sin^2\theta \left(m\frac{d\phi}{d\tau}\right)^2$$

$$-\frac{m^2\tilde{E}^2}{1 - \frac{2m}{r}} + \frac{m^2\left(\frac{dr}{d\tau}\right)^2}{1 - \frac{2m}{r}} + \frac{m^2\tilde{L}^2}{r^2} = -m^2$$

$$\boxed{\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) = \tilde{E}^2}$$

5

$$\boxed{\left(\frac{dr}{d\tau}\right)^2 + \tilde{V}^2(r) = \tilde{E}^2}$$

$r \gg M$ $\rightarrow$ $\left(\frac{dr}{d\tau}\right)^2 + 1 = \gamma^2 = \frac{1}{1-v^2} = 1 + v^2$

Earth: $r = 1\,\text{AU} = 1.5 \times 10^{13}$ cm.

$2M_\odot = 3\,\text{km} = 3 \times 10^5$ cm

$\tilde{L} = \frac{m\omega r^2}{m} = r \cdot \omega r = r \cdot \left(\frac{30\,\text{km/s}}{300{,}000\,\text{km/s}}\right) = \frac{10^{-4}}{1.5 \times 10^{9}\,\text{cm}}$

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photon $\quad p \cdot p = 0 \qquad p_t = -E \quad p_\phi = L$

$$-\frac{E^2}{1-\frac{2M}{r}} + \frac{(dr/d\lambda)^2}{1-\frac{2M}{r}} + \frac{L^2}{r^2} = 0$$

$$\boxed{\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right) = E}$$

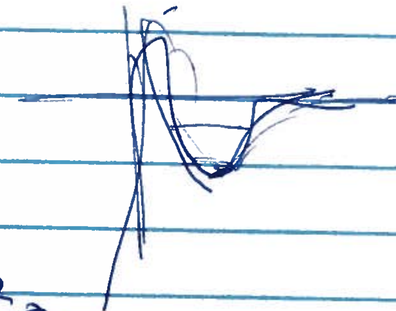
$r \to \infty$. $L = p_\phi = r^2 \frac{d\phi}{d\lambda} = p \cdot b = E \cdot b$

$$\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{E^2 b^2}{r^2}\left(1-\frac{2M}{r}\right)}_{\text{peak @ } \boxed{r=3M} \text{ (indep. of } E, b)} = E^2$$

Peak value $= \frac{E^2 b^2}{27M^2}$ $\qquad$ capture if $b^2 < 27M^2$

$$\boxed{\sigma = 27\pi M^2}$$

$$\frac{p^\phi}{p^t} = \frac{\frac{1}{r^2} L}{\frac{1}{(1-\frac{2m}{r})} E} = \frac{d\phi/d\lambda}{dt/d\lambda} = \frac{d\phi}{dt} = \Omega$$



$$\boxed{\Omega = \frac{(1-\frac{2m}{r})\tilde{L}}{r^2 \tilde{E}}}$$

circular orbit $\frac{dr}{dt} = 0$ $\left(1-\left(\frac{2m}{r^2}\right)\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) =$

$\frac{d^2 r}{dt^2} = 0$

$$\frac{d}{dr}\left[\left(1-\frac{2m}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\right] = \left(\frac{2m}{r^2}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) + \left(1-\frac{2m}{r}\right)\left(-2\frac{\tilde{L}^2}{r^3}\right) \Rightarrow 0$$

$$\rightarrow \boxed{L^2 = \frac{m r^2}{r-3m}} \longrightarrow \tilde{E} = \frac{(r-2m)^2}{r^2(r-3m)}$$

$$\Omega^2 = \frac{\left(1-\frac{2m}{r}\right)^2 \tilde{L}^2}{r^2 \tilde{E}} = \frac{\left(1-\frac{2m}{r}\right)^2 \left(\frac{m r^2}{\cancel{r-3m}}\right)}{\cancel{r^2}\frac{(r-2m)^2}{\cancel{r^2}(\cancel{r-3m})}} =$$

$$\boxed{\Omega^2 = \frac{m}{r^3}}$$

Kepler!

(Schwarzschild r.)