10|26|2016

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2$$

$$\rightarrow \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2m}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) = \tilde{E}^2$$

$\frac{\tilde{L}^2}{r^2} = 10^{-8}$

Earth: $r = 1\,AU = 1.5\times10^{13}$ cm

$\tilde{L} = \omega r^2 = rv = r\cdot\left(\frac{30\text{ km/s}}{300{,}000\text{ km/s}}\right) = 10^{-4}r$

$\frac{2\pi\cdot1.5\times10^{13}}{\pi\cdot10^7} = 3\cdot10^6$ cm/s $= 30$ km/s

$\frac{2M}{r} = \frac{3\text{ km}}{150\text{ M km}} = 2\times10^{-8}$

$r_{☿} = 0.387 \rightarrow 5\times10^{-7}$

$e^{2\alpha} = 1-\frac{2m}{r} \qquad \alpha = \frac{1}{2}\log\left(1-\frac{2m}{r}\right)$

$\boxed{R^{tr}{}_{tr} = \frac{2m}{r^3}}$ $\qquad$ $\boxed{R^{t\theta}{}_{t\theta} = R^{t\phi}{}_{t\phi} = -\frac{m}{r^3}}$

$\boxed{R^{r\theta}{}_{r\theta} = R^{r\phi}{}_{r\phi} = -\frac{m}{r^3}}$

No curvature singularity at $r=2m$

$\boxed{R^{\theta\phi}{}_{\theta\phi} = \frac{1}{r^2}\left(1-e^{-2\beta}\right) = \frac{1}{r^2}\left(1-\left(1-\frac{2m}{r}\right)\right) = \frac{2m}{r^3}}$

$\boxed{R^{t\theta}{}_{r\theta} = R^{t\phi}{}_{r\phi} = -\frac{m}{r^3}}$

$$\frac{p^\phi}{p^t} = \frac{\frac{1}{r^2}L}{\frac{1}{(1-\frac{2m}{r})}E} = \frac{\mu\, d\phi/d\lambda}{\mu\, dt/d\lambda} = \frac{d\phi}{dt} = \Omega$$

$$\boxed{\Omega = \frac{(1-\frac{2m}{r})\tilde{L}}{r^2 \tilde{E}}}$$

circular orbit $\frac{dr}{dt}=0$ $\frac{d^2r}{dt^2}$ $\cancel{\left(1-\left(\frac{2m}{r^2}\right)\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)} =$

$$\frac{d}{dr}\left[\left(1-\frac{2m}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\right] = \left(\frac{2m}{r^2}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) + \left(1-\frac{2m}{r}\right)\left(-2\frac{\tilde{L}^2}{r^3}\right) = 0$$

$$\rightarrow \boxed{\tilde{L}^2 = \frac{mr^2}{r-3m}} \longrightarrow \tilde{E} = \frac{(r-2m)^2}{r^2(r-3m)}$$

$$\Omega^2 = \frac{\left(1-\frac{2m}{r}\right)^2 \tilde{L}^2}{r^2\tilde{E}} = \frac{\left(1-\frac{2m}{r}\right)^2\left(\frac{mr^2}{r-3m}\right)}{r^2\frac{(r-2m)^2}{r^2(r-3m)}} =$$

$$\boxed{\Omega^2 = \frac{m}{r^3}}$$

Kepler!

(Schwarzschild r.)

$\frac{GM}{r^3}$

②

Isotropic coordinates

$$ds^2 = \ldots + F^2(\bar r)\left(d\bar r^2 + \bar r^2 d\omega^2\right)$$

$$r^2 = \bar r^2 F^2(\bar r) \qquad \boxed{r = \bar r F(\bar r)}$$

$$dr = \left(\bar r \frac{dF}{d\bar r} + F\right) d\bar r$$

$$\frac{dr^2}{1-\frac{2m}{r}} = \frac{\left(\bar r \frac{dF}{d\bar r} + F\right)^2 d\bar r^2}{1 - \frac{2m}{F\bar r}} = F^2 d\bar r^2$$

$$\boxed{\left(\bar r \frac{dF}{d\bar r} + F\right)^2 = F^2 - \frac{2m}{\bar r} F}$$

$\bar r \to \infty$ $\quad F \to 1 \qquad F = 1 + \frac{A}{\bar r} + \frac{B}{\bar r^2}$

$$\bar r \frac{dF}{d\bar r} + F = \cancel{-\frac{A}{\bar r}} - \frac{2B}{\bar r^2} + 1 + \cancel{\frac{A}{\bar r}} + \frac{B}{\bar r^2} = 1 - \frac{B}{\bar r^2}$$

$$\left(1 - \frac{B}{\bar r^2}\right)^2 = 1 - \frac{2B}{\bar r^2} + \frac{B^2}{\bar r^4}$$

$$= \left(1 + \frac{2A}{\bar r} + \frac{2B}{\bar r^2} + \frac{A^2}{\bar r^2} + \frac{B^2}{\bar r^4} + \frac{2AB}{\bar r^3}\right)$$

$$- \frac{2m}{\bar r}\left(1 + \frac{A}{\bar r} + \frac{B}{\bar r^2}\right)$$

$$\cancel{1} - \frac{2B}{\bar r^2} + \cancel{\frac{B^2}{\bar r^4}} = \cancel{1} + \cancel{\frac{2(A-m)}{\bar r}} + \frac{A^2 + 2B - 2Am}{\bar r^2}$$

$$+ \cancel{\frac{2AB - 2Bm}{\bar r^3}} + \cancel{\frac{B^4}{\bar r^4}}$$

$A = m$ $\qquad -2B = 2B - m^2 \qquad 4B = m^2$

$$F = 1 + \frac{m}{\bar r} + \frac{m^2}{4\bar r^2} = \left(1 + \frac{m}{2\bar r}\right)^2$$

$$r = \bar r F = \bar r \left(1 + \frac{m}{2\bar r}\right)^2$$

$$1 - \frac{2m}{r} = 1 - \frac{2m}{\bar r \left(1 + \frac{m}{2\bar r}\right)^2} = \frac{\left(1 + \frac{m}{2\bar r}\right)^2 - \frac{2m}{\bar r}}{\left(1 + \frac{m}{2\bar r}\right)^2}$$

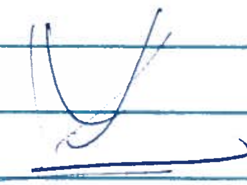
$$= \left(\frac{1 - \frac{m}{2\bar r}}{1 + \frac{m}{2\bar r}}\right)^2$$

$$\boxed{ds^2 = -\left(\frac{1 - \frac{m}{2\bar r}}{1 + \frac{m}{2\bar r}}\right)^2 dt^2 + \left(1 + \frac{m}{2\bar r}\right)^2 \left(d\bar r^2 + \bar r^2 d\Omega^2\right)}$$

$\bar r > 2m$: $\quad \left(1 - \frac{m}{2\bar r}\right)^2 \left(\left(1 - \frac{m}{2\bar r}\right)^2 : 1 - \frac{2m}{\bar r} = (1 + 2\phi)$

$$\left(1 + \frac{m}{2\bar r}\right)^2 = 1 + \frac{2m}{\bar r} = (1 - 2\phi)$$

$$r = \bar{r}\left(1+\frac{M}{2\bar{r}}\right)^2 = \bar{r} + M + \frac{M^2}{4\bar{r}}$$



min @ $\bar{r} = \frac{M}{2}$ $\quad r = 2M \quad$ $r \geq 2M$ only.

Precession:

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) = \tilde{E}^2$$

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\cdot\frac{d\phi}{d\tau} = \left(\frac{dr}{d\phi}\right)\left(\frac{1}{r^2}\tilde{L}\right) = -\tilde{L}\frac{d}{d\phi}\left(\frac{1}{r}\right)$$

Let $u = \frac{1}{r}$

$$\tilde{L}^2\left(\frac{du}{d\phi}\right)^2 + (1-2Mu)(1+\tilde{L}^2u^2)$$

$$= \tilde{L}^2u'^2 + 1 - 2Mu + \tilde{L}^2u^2 - 2M\tilde{L}^2u^3 = \tilde{E}^2$$

$\frac{d}{d\phi}$: $$\tilde{L}^2\cdot\cancel{2}\cancel{u'}u'' - \cancel{2}M\cancel{u'} + \cancel{2}\tilde{L}^2u\cancel{u'} - \overset{3}{\cancel{6}}M\tilde{L}^2\cancel{u'}u^2 = 0$$

$$\boxed{u'' + u = \frac{M}{\tilde{L}^2} + 3Mu^2}$$

Circular orbit $\quad u = u_0 = \text{constant}$

$$u_0 = \frac{M}{L^2} + 3Mu_0^2 \simeq \frac{M}{L^2} + 3M\left(\frac{M^2}{L^4}\right)$$

$$\cong \frac{M}{L^2}\left(1 + \frac{3M^2}{L^2}\right)$$

$M = 1.5\text{ km}$

$L = (\ )10^9\text{ km}$

perturbed: $\quad u = u_0 + \Delta u$

$$\frac{d^2}{d\phi^2}\Delta u + u_0 + \Delta u = \frac{m}{L^2} + 3m(u_0 + \Delta u)^2$$

$$\frac{d^2}{d\phi^2}(\Delta u) + \Delta u = 6mu_0\,\Delta u + \cancel{3m(\Delta u)^2}$$

$$\frac{d^2}{d\phi^2}(\Delta u) + (1 - 6mu_0)\,\Delta u = 0$$

$$\boxed{\Delta u = u_1 \cos \jmath\phi} \qquad \jmath^2 = 1 - 6mu_0 \doteq (1 - 3mu_0)^2$$

return to perihelion $\quad \jmath\phi = 2\pi$

$$\phi = \frac{2\pi}{\jmath} = 2\pi(1 + 3mu_0)$$

$$\frac{\Delta\phi}{2\pi} = 3mu_0 \quad \text{per orbit}$$

Elliptical orbit.

$$\boxed{u = \frac{1+e\cos\phi}{a(1-e^2)}}$$

$a = 0.38709$ AU

$e = 0.2056\ldots$

$= u_0 + (u_1 \cos\phi)$ $\qquad u_0 = \frac{1}{a(1-e^2)}$

$$3 u_0 = 3\left(\frac{Gm}{c^2}\right)\left(\frac{1}{a(1-e^2)}\right)$$

$$= \frac{3(1.476\text{ km})}{(57.91 \text{ Mkm})(1-.04227)} = \underline{7.99\times10^{-8}}$$

$T = \underline{87.97 d} \rightarrow 100 Y = \underline{415.2 \text{ orbits}}$

$$(415.2)(7.99\times10^{-8})(2\pi) = \underline{42.9 \text{ arcsec}}$$

$\underline{PSR 1913+16}$. $\qquad P = \underline{7.75 h}$.

precession. $\qquad \dot{\omega} = 4.2263 \text{ deg/yr}$ $\qquad$ 1130.78 orbits.

6.523×10^{-5} per orbit