10/31/2016

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2$$

$$\frac{\partial g_{\mu\nu}}{\partial \phi} = 0 \rightarrow p_\phi = \text{constant} = \tilde{L}$$

All directions K^μ = "Killing vector"

$$\boxed{K_{\mu;\nu} + K_{\nu;\mu} = 0}$$

$$\boxed{\nabla_{(\nu} K_{\mu)} = 0}$$

$x^\mu(\lambda)$ = geodesic. $\frac{dx^\mu}{d\lambda} = p^\mu$.

$$\rightarrow \nabla_p (K \cdot p) = p^\nu \nabla_\nu (K_\mu p^\mu)$$

$$= p^\nu (K_{\mu;\nu} p^\mu + K_\mu p^\mu{}_{;\nu})$$

$$= \cancel{K_{(\mu;\nu)}} p^\mu p^\nu + K_\mu (\cancel{p^\nu p^\mu{}_{;\nu}}) = 0$$

2

$$\vec{L} = \vec{r} \times \vec{p} = \hat{x}(yp_z - zp_y) + \hat{y}(zp_x - xp_z) + \hat{z}(xp_y - yp_x)$$

$$R^i = (-y, x, 0)$$
$$S^i = (z, 0, -x)$$
$$T^i = (0, -z, y)$$

$$L_z = \frac{\partial}{\partial\phi}$$

$$\det J_{ij} = r^2 \sin\theta$$

Spherical coord

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\left(\frac{\partial x^i}{\partial x'^j}\right) = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$\left(\frac{\partial x'^i}{\partial x^j}\right) = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\phi & \frac{1}{r}\cos\theta\sin\phi & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix}$$

$$R^r = \left(\frac{\partial r}{\partial x^i}\right) R^i = 0$$
$$R^\theta = \left(\frac{\partial \theta}{\partial x^i}\right) R^i = 0$$
$$R^\phi = \left(\frac{\partial \phi}{\partial x^i}\right) R^i = 1$$

$$\hat{R} = \hat{e}_\phi$$

$$\hat{S} = \cos\phi\, \hat{e}_\theta - \frac{\cos\theta}{\sin\theta}\sin\phi\, \hat{e}_\phi$$

$$\hat{T} = -\sin\theta\, \hat{e}_\phi - \frac{\cos\theta}{\sin\theta}\cos\phi\, \hat{e}_\phi$$

R, S, T: all killing vectors

$[R, S] = T$ (et cyc.)

← Any spherical geometry,

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

T. $T^i = \begin{pmatrix} -\sin\phi \\ -\cot\theta \cos\phi \end{pmatrix}$

$$T_i = \begin{pmatrix} -a^2 \sin\phi \\ -a^2 \sin\theta \cos\theta \cos\phi \end{pmatrix}$$

$$T_{\theta;\phi} = \frac{\partial T_\theta}{\partial \phi} - \Gamma^i_{\theta\phi} T_i = \frac{\partial T_\theta}{\partial \phi} - \Gamma^\phi_{\theta\phi} T_\phi$$

$$= \frac{\partial}{\partial\phi}(-a^2 \sin\phi) + \left(\frac{\cos\theta}{\cancel{\sin\theta}}\right)\left(+a^2 \cancel{\sin\theta} \cos\theta \cos\phi\right)$$

$$= -a^2\cos\phi + a^2 \cos^2\theta \cos\phi = -a^2 \sin^2\theta \cos\phi$$

$$T_{\phi;\theta} = \frac{\partial T_\phi}{\partial \theta} - \Gamma^i_{\phi\theta} T_i = \frac{\partial T_\phi}{\partial \theta} - \Gamma^\phi_{\phi\theta} T_\phi$$

$$= \frac{\partial}{\partial\theta}(-a \sin\theta \cos\theta \cos\phi) - \left(\frac{\cos\theta}{\cancel{\sin\theta}}\right)\left(-a^2 \cancel{\sin\theta} \cos\theta \cos\phi\right)$$

$$= -a^2 \cos\phi(\cancel{\cos^2\theta} - \sin^2\theta) + a^2 \cancel{\cos^2\theta} \cos\phi$$

$$= +a^2 \sin^2\theta \cos\phi$$

$$T_{\theta;\theta} = \cancel{\frac{\partial T_\theta}{\partial\theta}} - \cancel{\Gamma^i_{\theta\theta} T_i} = 0 \qquad T_{\phi;\phi} = 0$$

(4).

$$[R,S] = \nabla_R S - \nabla_S R \qquad R^i = \begin{pmatrix}0\\1\end{pmatrix}$$

$$= R^j S^i{}_{;j} - S^j R^i{}_{;j} = R^j S^i{}_{,j} - \cancel{S^j R^i{}_{,j}}$$

$$= \frac{\partial S^i}{\partial\phi} = \frac{\partial}{\partial\phi}\begin{pmatrix}\cos\phi \\ -\frac{\cos\theta}{\sin\theta}\sin\phi\end{pmatrix} = \begin{pmatrix}-\sin\phi \\ -\frac{\cos\theta}{\sin\theta}\cos\phi\end{pmatrix} = T$$

$$\boxed{[R,S] = T}$$ et. cyc.

$$[\nabla_\alpha,\nabla_\beta] K^\mu = R^\mu{}_{\nu\alpha\beta} K^\nu \qquad (\text{def. } R^\mu{}_{\nu\alpha\beta}) \quad (3.113)$$

$$K_{\sigma;\rho\mu} - K_{\sigma;\mu\rho} = R_\sigma{}^\lambda{}_{\mu\rho} K_\lambda = R^\lambda{}_{\sigma\rho\mu} K_\lambda$$

permute:
$$K_{\sigma;\rho\mu} - K_{\sigma;\mu\rho} + K_{\rho;\mu\sigma} - K_{\rho;\sigma\mu} + K_{\mu;\sigma\rho} - K_{\mu;\rho\sigma}$$

$$= R^\lambda{}_{\sigma\rho\mu} + R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\mu\rho\sigma} = 0.$$

$$\underbrace{K_{\sigma;\rho\mu} - K_{\sigma;\mu\rho}}_{RK} - K_{\mu;\rho\sigma} = 0$$

$$\boxed{K_{\mu;\rho\sigma} = R^\lambda{}_{\sigma\rho\mu} K_\lambda} \qquad \text{cf. } (3.176)$$

$$\nabla_\mu \nabla_\sigma K^\rho = R_{\lambda\mu\sigma}{}^\rho K^\lambda = R^\rho{}_{\sigma\mu\lambda} K^\lambda$$

$$\boxed{K^\mu{}_{;\rho\sigma} = R_{\lambda\sigma\rho}{}^{\mu} K^\lambda = R^\mu{}_{\rho\sigma\lambda} K^\lambda} \quad (3.176)$$

Tr. $\mu\sigma$

$$\boxed{K^\mu{}_{;\rho\mu} = R^\mu{}_{\rho\mu\lambda} K^\lambda = R_{\rho\lambda} K^\lambda} \quad (3.177)$$

$$\boxed{\Box^2 K^\rho + R^\rho{}_\mu K^\mu = 0}$$

Vacuum $(R=0)$

$\longleftrightarrow$ EOM solution

$$K^\mu{}_{;\rho}{}^{\sigma} = R^\mu{}_{\rho}{}^{\sigma}{}_{\lambda} K^\lambda = -R_\rho{}^{\mu\sigma}{}_\lambda K^\lambda$$

$$K^\mu{}_{;\rho}{}^{\rho} = -R^\mu{}_\lambda K^\lambda \qquad \boxed{\Box^2 K^\mu + R^\mu{}_\lambda K^\lambda = 0}$$

Bianchi identity $\quad R_{\rho\sigma[\lambda\mu;\nu]} = 0$

$$\nabla^\mu R_{\rho\mu} = R_{\rho\mu}{}^{;\mu} = \frac{1}{2}\nabla_\rho R$$

$$\nabla_\mu(\Box^2 K^\mu) + \nabla_\mu(R^\mu{}_\lambda K^\lambda) = 0$$

$$\hookrightarrow K^\lambda(R^\mu{}_{\lambda;\mu}) + \underbrace{R^{\mu\lambda}\nabla_\mu K_\lambda}_{=0}$$

$$= K^\lambda\left(\frac{1}{2}\nabla_\lambda R\right) = \left(\frac{1}{2}\right)\nabla_K R = 0$$

Bianchi, contracted $\longrightarrow$ $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$

$$\frac{1}{2}K^\nu \nabla_\nu R = K^\nu \cancel{\nabla_\mu R^\mu_\nu} = K^\nu \nabla^\mu R_{\mu\nu}$$

$$= \nabla^\mu (K^\nu R_{\mu\nu}) - R_{\mu\nu}\nabla^\mu K^\nu$$

$$R_{\mu\nu}K^\nu = \nabla_\nu \nabla_\mu K^\nu \qquad (3.177)$$

$$\nabla^\mu (K^\nu R_{\mu\nu}) = \nabla^\mu \nabla^\nu \nabla_\mu K_\nu$$

$$= \nabla^\mu \nabla^\nu \nabla_{[\mu}K_{\nu]}$$

$$= \nabla^{[\mu}\nabla^{\nu]} \nabla_{[\mu}K_{\nu]}$$

$$= \frac{1}{2}(\nabla^\mu\nabla^\nu - \nabla^\nu\nabla^\mu)\nabla_{[\mu}K_{\nu]} = 0.$$

$$= \frac{1}{2}\left(R_{[\lambda\nu]}\nabla^{[\lambda}K^{\nu]} - R_{[\lambda\mu]}\nabla^{[\mu}K^{\lambda]}\right)$$

$$[\nabla_\rho, \nabla_\sigma]T^{\mu\nu} = R^\mu{}_{\lambda\rho\sigma}T^{\lambda\nu} + R^\nu{}_{\lambda\rho\sigma}T^{\mu\lambda}$$

$$[\nabla_\rho, \nabla_\sigma]T^{\rho\sigma} = R^\rho{}_{\lambda\rho\sigma}T^{\lambda\sigma} + R^\sigma{}_{\lambda\rho\sigma}T^{\rho\lambda}$$

$$= R_{\lambda\sigma}T^{\lambda\sigma} - R_{\lambda\rho}T^{\rho\lambda}$$