11/2/2016

$\boxed{\text{Lie derivative}}$ $\mathcal{L}$ directional

$$\mathcal{L}_u f = \nabla_u f = u^\alpha \nabla_\alpha f. \quad \text{(scalar)}$$

$$\mathcal{L}_u v = [u, v] = \nabla_u v - \nabla_v u. \quad \text{(vector)}$$

$$= u^\alpha v^\mu{}_{;\alpha} - v^\alpha u^\mu{}_{;\alpha} = u^\alpha v^\mu{}_{,\alpha} - v^\alpha u^\mu{}_{,\alpha}$$

$$\mathcal{L}_u (fv) = \nabla_u (fv) - \nabla_{(fv)} u$$

$$= (\nabla_u f) \cdot v + f(\nabla_u v) - f(\nabla_v u)$$

$$= (\nabla_u f) v + f \cdot (\nabla_u v - \nabla_v u)$$

$$= \mathcal{L}_u f \cdot v + f \cdot \mathcal{L}_u v \quad \text{leibnitz rule}$$

$$\mathcal{L}_u \langle w, v \rangle = \mathcal{L}_u (w_\alpha v^\alpha) = \langle \mathcal{L}_u w, v \rangle + \langle w, \mathcal{L}_u v \rangle$$

$$= \nabla_u \langle w, v \rangle = \langle \nabla_u w, v \rangle + \langle w, \nabla_u v \rangle$$

$$\langle \mathcal{L}_u w, v \rangle = \langle \nabla_u w, v \rangle + \cancel{\langle w, \nabla_u v \rangle} - \langle w, \cancel{\nabla_u v} - \nabla_v u \rangle$$

$$\langle \mathcal{L}_u w, v \rangle = (\nabla_u w)_\alpha v^\alpha + w_\alpha (v^\beta \cdot \nabla_\beta u^\alpha)$$

$$\Rightarrow \boxed{(\mathcal{L}_u w)_\mu = u^\alpha \nabla_\alpha w_\mu + w_\alpha \nabla_\mu u^\alpha}$$

②

covariant $\cdot$ $(\mathcal{L}_u w)_\mu = u^\alpha w_{\mu;\alpha} + w_\alpha u^\alpha{}_{;\mu}$

$$= u^\alpha (w_{\mu,\alpha} - \Gamma^\lambda_{\mu\alpha} w_\lambda) + w_\alpha (u^\alpha{}_{,\mu} + \Gamma^\alpha_{\lambda\mu} u^\lambda)$$

$$= u^\alpha w_{\mu,\alpha} + w_\alpha u^\alpha{}_{,\mu} - \cancel{\Gamma^\lambda_{\mu\alpha} w_\lambda u^\alpha} + \cancel{\Gamma^\alpha_{\lambda\mu} w_\alpha u^\lambda}$$

($\mathcal{L}$ defined w/ w/o covariant ~~ge~~ ∇_μ)

Second rank tensor . each index

$$\mathcal{L}_u T_{\mu\nu} = u^\sigma T_{\mu\nu;\sigma} + u^\sigma{}_{;\mu} T_{\sigma\nu} + u^\sigma{}_{;\nu} T_{\mu\sigma}.$$

metric

$$\mathcal{L}_u g_{\mu\nu} = u^\sigma \cancel{g_{\mu\nu;\sigma}} + u^\sigma{}_{;\mu} g_{\sigma\nu} + u^\sigma{}_{;\nu} g_{\mu\sigma}$$

$$\boxed{\mathcal{L}_u g_{\mu\nu} = u_{\mu;\nu} + u_{\nu;\mu}}$$

K_μ = killing vector $\boxed{\mathcal{L}_K g_{\mu\nu} = 0}$

$$\boxed{K^\mu \nabla_\mu R = 0}$$

Inner workings of Lie derivative.

Change of coordinate

(A) Move to new (shifted) point.
(B) Change name of coordinate at same point.

Scalar: $x'^{\mu} = x^{\mu} + \varepsilon u^{\mu}$

$$\phi' = \phi(x') = \phi(x + \varepsilon u) = \phi(x) + \varepsilon u^{\alpha} \frac{\partial \phi}{\partial x^{\alpha}}$$

$$\lim_{\varepsilon \to 0} \frac{\phi' - \phi}{\varepsilon} = u^{\alpha} \frac{\partial \phi}{\partial x^{\alpha}} = \nabla_u \phi$$

vector $V'^{\mu} = V^{\mu} + \frac{\partial V^{\mu}}{\partial x^{\alpha}} \cdot \varepsilon u^{\alpha}$

change coordinate value. $x''^{\mu} = x'^{\mu} - \varepsilon u^{\mu}$.

$$V''^{\mu} = (V'^{\alpha}) \left(\frac{\partial x''^{\mu}}{\partial x'^{\alpha}} \right) = V'^{\alpha} \left(\delta^{\mu}_{\alpha} - \varepsilon \frac{\partial u^{\mu}}{\partial x^{\alpha}} \right)$$

$$= \left(V^{\alpha} + \frac{\partial V^{\alpha}}{\partial x^{\beta}} \cdot \varepsilon u^{\beta} \right) \left(\delta^{\mu}_{\alpha} - \varepsilon \frac{\partial u^{\mu}}{\partial x^{\alpha}} \right)$$

$$= V^{\mu} + \frac{\partial V^{\mu}}{\partial x^{\beta}} \cdot \varepsilon u^{\beta} - V^{\alpha} \cdot \varepsilon \frac{\partial u^{\mu}}{\partial x^{\alpha}}$$

$$\lim_{\varepsilon \to 0} \frac{V'' - V}{\varepsilon} = u^{\alpha} \frac{\partial V^{\mu}}{\partial x^{\alpha}} - V^{\alpha} \frac{\partial u^{\mu}}{\partial x^{\alpha}} = [u, v]$$

Schwarzschild revisited. $\left\{ \begin{array}{l} \text{uniqueness.} \\ \text{sources} \end{array} \right.$

spherical symmetry. $(\theta, \phi.)$

pitch, yaw, roll

$R = \frac{\partial}{\partial \phi}$ $\quad S = \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}$

$T = -\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi}.$

$[K_i, K_j] = \varepsilon_{ijk} K_k$

Integral curves $\left(\frac{dx^i}{d\lambda} = K^i \right)$ → trace out sphere. ("submanifold")

4D spacetime "subfoliated" into spheres.

$$ds^2 = \underline{d\sigma^2(a,b)} + r^2(a,b)\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

[indep. of θ, ϕ (else depends on direction)

[$\cancel{r^2}(a,b) = \cancel{r^2}(a_0, b_0)\, d\Omega^2$ else "lumpy"

$$ds^2 = g_{aa}\, da^2 + g_{ab}(da\, db + db\, da) + g_{bb}\, db^2 + r^2 d\Omega^2$$

$$= g_{tt}\, dt^2 + g_{rt}(dt\, dr + dr\, dt) + g_{rr}\, dr^2$$

$r(a,b)$ inverted to $b(a,r)$: trade $b \Leftrightarrow r$ good.

Can choose $t'(t)$ so that there is no $dr\,dt'$.

$$dt' = \frac{\partial t'}{\partial r} dr + \frac{\partial t'}{\partial t} dt.$$

$$g'_{tt}\, dt'^2 + \cancel{\delta r^2} + g'_{rr}\, dr^2 \qquad \text{(want)}.$$

$$= g'_{tt}\left[\left(\frac{\partial t'}{\partial t}\right)^2 \underline{dt^2} + \left(\frac{\partial t'}{\partial t}\right)\left(\frac{\partial t'}{\partial r}\right) \underline{\left(dr\,dt + dt\,dr\right)} + \left(\frac{\partial t'}{\partial r}\right)^2 dr^2\right] + \underline{g'^2_{rr}\, dr^2}$$

$$= \underline{g_{tt}}\, dt^2 + \underline{g_{rt}}\left(dr\,dt + dt\,dr\right) + \underline{g_{rr}}\, dr^2$$

$$\underline{g'_{tt}\left(\frac{\partial t'}{\partial t}\right)^2 = g_{tt}}$$

$$\underline{g'_{tt}\left(\frac{\partial t'}{\partial t}\right)\left(\frac{\partial t'}{\partial r}\right) = g_{tr}}$$

$$\underline{g'_{tt}\left(\frac{\partial t'}{\partial r}\right)^2 + g'_{rr}} = \underline{\underline{g_{rr}}}$$

3 conditions

3 functions

g'_{tt}

g'_{rr}

$t(a,r)$