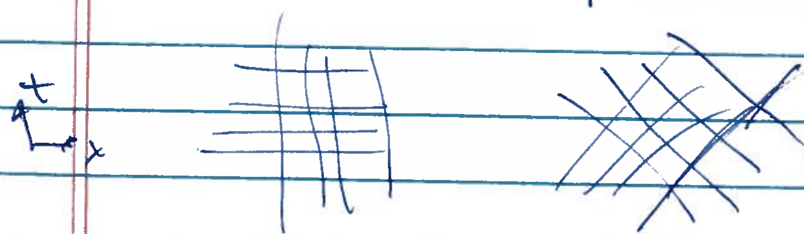
11/4/2016 metric $g_{\mu\nu}$ symmetric, can always be diagonalized.

$$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

$$ds^2 = -dt^2 + dx^2 = -(dt-dx)(dt+dx)$$

let $\left.\begin{aligned} u &= t-x \\ v &= t+x \end{aligned}\right\}$ $ds^2 = -du\,dv$ $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$



u = constant (right →)

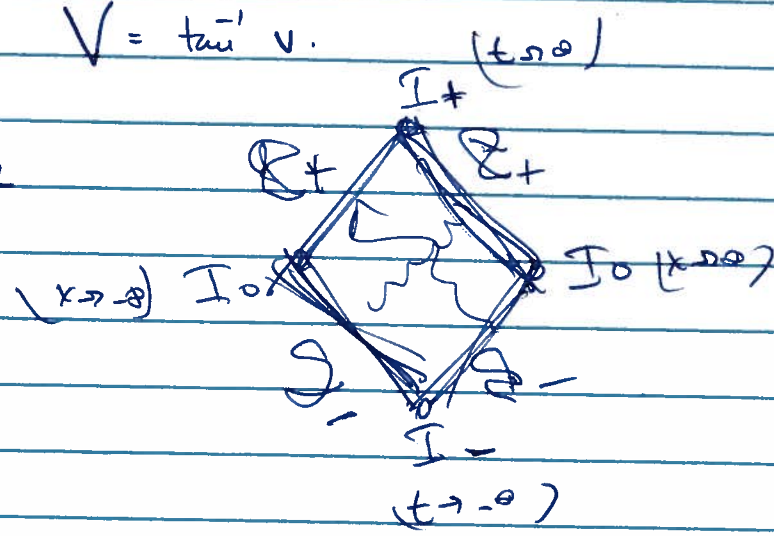
v = constant (left ←)

$-\infty < u, v < \infty$.

$U = \tan^{-1} u$

$V = \tan^{-1} v$.

$-\frac{\pi}{2} < U, V < \frac{\pi}{2}$



All of spacetime

Penrose

②

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2$$

$\Gamma^t_{tt} = \dot{\alpha}$ $\quad$ $\Gamma^t_{rr} = e^{2(\beta-\alpha)}\dot{\beta}$ $\quad$ $\dot{} = \frac{\partial}{\partial t}$

$\Gamma^r_{tr} = \dot{\beta}$ $\quad$ $\Gamma^r_{rr} = \beta'$ $\quad$ $' = \frac{\partial}{\partial r}$

$\Gamma^r_{tt} = e^{2(\alpha-\beta)}\alpha'$ $\quad$ $\cancel{\Gamma^r_{tr} = \dot{\beta}}$

$\Gamma^r_{\phi\phi} = -r\sin^2\theta\, e^{-2\beta}$ $\quad$ $\Gamma^r_{\theta\theta} = -r e^{-2\beta}$

$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}$

$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ $\quad$ $\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

$$R^{tr}{}_{tr} = e^{-2\alpha}\left(\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}\right) + e^{-2\beta}\left(\alpha'\beta' - \alpha'' - \alpha'^2\right)$$ (new)

$$R^{t\theta}{}_{t\theta} = R^{t\phi}{}_{t\phi} = -\frac{1}{r}e^{-2\beta}\alpha'$$ (same)

$$R^{t\theta}{}_{r\theta} = R^{t\phi}{}_{r\phi} = -\frac{1}{r}e^{-2\alpha}\dot{\beta}$$ (new)

$$R^{r\theta}{}_{r\theta} = R^{r\phi}{}_{r\phi} = \frac{1}{r} e^{-2\beta} \beta'$$

$$R^{\theta\phi}{}_{\theta\phi} = \frac{1}{r^2}\left(1 - e^{-2\beta}\right)$$

(same).

$$R^t{}_t = e^{-2\alpha}\left(\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}\right) + e^{-2\beta}\left(\alpha'' + \alpha'^2 + \frac{2\alpha'}{r} - \alpha'\beta'\right)$$

$$R^r{}_r = -e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' - \frac{2\beta'}{r}\right) + e^{-2\alpha}\left(\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}\right)$$

$$R^\theta{}_\theta = R^\phi{}_\phi = \frac{1}{r^2} e^{-2\beta}\left(r\left(\beta' - \alpha'\right) - 1\right) + \frac{1}{r^2}$$

$$\boxed{R^t{}_r = \frac{2}{r}\dot{\beta}\, e^{-2\alpha}}$$ New

Vacuum: $R_{\mu\nu} = 0$. $\quad R_{tr} = 0 \Rightarrow \dot{\beta} = 0$

$$R^\theta{}_\theta = \frac{e^{-2\beta}}{r^2}\left(r(\beta' - \alpha') - 1\right) + \frac{1}{r^2}$$

$$\frac{\partial}{\partial t} R^\theta{}_\theta = \frac{e^{-2\beta}}{r^2}\, r\left(\cancel{\dot{\beta}'} - \dot{\alpha}'\right) \quad \dot{\alpha}' = 0$$

$\beta = \beta(r)$ $\quad\quad \alpha = f(r) + g(t)$

$$g_{tt}\, dt^2 = e^{2f(r)}\, e^{2g(t)}\, dt^2 = e^{2f(r)}\, dt'^2$$

$$e^{2g(t)}\, dt^2 = dt'^2 \qquad \frac{dt'}{dt} = g(t)$$

Back to $\quad ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2$

Spherically symmetric Vacuum solution has a time-like killing vector $\left(\frac{\partial g}{\partial t} = 0\right)$

Sources $\quad T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$ $\qquad \frac{\partial T}{\partial t} = 0$

$R_{tr} = 0 \rightarrow \dot{\beta} \equiv 0 \qquad \beta = \beta(r)$

$\frac{\partial}{\partial t}(R^\theta_\theta) = 0 \rightarrow \dot{\alpha}' = 0$ $\quad$ as before.

Timelike $K_t \rightarrow$ "stationary"

Hypersurface orthogonal $\rightarrow$ "static"
(stationary current is stationary)

$$G^t{}_t = \frac{1}{r^2} e^{-2\beta}\left(2r\beta' + e^{2\beta} - 1\right) = 8\pi G\rho.$$

$$G^r{}_r = \frac{1}{r^2}\left(2r\alpha' + 1 - e^{+2\beta}\right) = 8\pi G p.$$

$$G^\theta{}_\theta = G^\phi{}_\phi = e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha' - \beta'}{r}\right) = 8\pi G p$$

Let $\boxed{1 - e^{-2\beta} = \frac{2Gm(r)}{r}}$ $\boxed{e^{-2\beta} = 1 - \frac{2Gm(r)}{r}}$

$$r\left(1 - e^{-2\beta}\right) = 2Gm$$

$\frac{d}{dr} \rightarrow$ $$1 - e^{-2\beta} + 2r\beta' e^{-2\beta} = 2Gm'$$

$$e^{-2\beta}\left(2r\beta' + e^{2\beta} - 1\right) = 2Gm'$$

$G^t{}_t \rightarrow$ $$8\pi r^2 G\rho = 2Gm'.$$

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho}$$

$$\boxed{m(r) = m_0 + \int_0^r 4\pi r^2 dr\, \rho(r)}$$

$$d^3V = \sqrt{g}\, d^3x = \frac{4\pi r^2 dr}{\sqrt{1-\frac{2m}{r}}}$$

$$\boxed{\bar{m} = \int \frac{4\pi r^2 dr\, \rho(r)}{\sqrt{1-\frac{2m}{r}}} > m}$$

$\bar{m} - m = E_B.$

$G^r_r.$

$$\frac{1}{r^2}\left(2r\alpha' + 1 - \frac{1}{1-\frac{2m}{r}}\right) = 8\pi G p.$$

$$\boxed{\frac{d\alpha}{dr} = \frac{G\left[m(r) + 4\pi r^3 p\right]}{r\left(r - 2m(r)\right)}}$$

$T^{\mu\nu}{}_{;\nu} = 0$

$$\left[(\rho+p)u^\mu u^\nu + p g^{\mu\nu}\right]_{;\nu}$$

$$= (\rho+p)_{,\nu} u^\mu u^\nu + (\rho+p) u^\mu{}_{;\nu} u^\nu$$

$$+ (\rho+p) u^\mu u^\nu{}_{;\nu} + p^{,\mu} = 0.$$

$$u^\mu \left[\frac{d}{d\tau}(\rho+p) + \theta(\rho+p)\right] + p^{,\mu} = 0$$

$r=r$. $u^r = 0$ $\quad u^t = e^{-\alpha}$ $\quad (u^t)^2(-e^{2\alpha}) = -1$.

$$(\rho+p)\,\Gamma_{r\alpha\beta}\,u^\alpha u^\beta = -p_{,r} = -\frac{\partial p}{\partial r}$$

$$\Gamma_{rtt} = \frac{1}{2}\left(\cancel{g_{rt,t}} + \cancel{g_{rt,t}} - g_{tt,r}\right)$$

$$= \frac{1}{2}(-)(2\alpha')\,e^{+2\alpha} = -\alpha' e^{2\alpha}$$

$$\boxed{(\rho+p)\frac{\partial\alpha}{\partial r} = -\frac{\partial p}{\partial r}}$$

$$\boxed{\frac{\partial p}{\partial r} = -\frac{G(\rho+p)\left[M(r) + 4\pi r^3 p(r)\right]}{r(r-2M)}}$$

TOV

Non-relativistic $p \ll \rho$. $M \ll r$.

$$\boxed{\frac{\partial p}{\partial r} = -\frac{GM}{r^2}\rho}$$

pressure [out]
[$\to$ gravity [in]]

Every extra factor makes gravity stronger